Mathemat^l. vol 1.

THE

The Mathematician.

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NUMBER I.

Containing a

DISSERTATION

ON THE

RISE, PROGRESS, and IMPROVEMENT of

GEOMETRY:

ALSO,

The chief Properties of the PARABOLA, demonſtrated after an eaſy, ſhort, and elegant Manner.

Together with

A Collection of Twenty Choice PROBLEMS, in Algebra, Geometry, and Mathematical Philoſophy, the greater Part whereof, never before publiſhed.

By a SOCIETY.

Magna eſt Veritas et prævalebit.

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G.R. III

THE

PREFACE.

SINCE Cuſtom has made it faſhionable to introduce moſt new Publications with a Preface, giving ſome Account of the Performance, as a Prologue to a Play, or a Bill of Fare to a Feaſt; the Editors of this Work have, therefore, thought it incumbent on them to acquaint their Readers with the Reaſons which induced them to undertake a Deſign, the firſt Number whereof is contained in the following Sheets.

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It is juſtly obſerved, that the preſent Age is remarkable for its Attachment to Eſſay or Index-Learning; which is a compendious and eaſy Way of acquiring a moderate Skill in Science, to the Content of many, without the Trouble of ſtudying Syſtems, and examining their fundamental Principles. Such a Diſpoſition the immortal Pope *has beautifully deſcribed in theſe Words.*

Now Index-Learning turns no Student pale,
Yet holds the Eel of Science by the Tail.

The Preſs ſcarce ever, now-a-Days, teems with Corpus's, Codexes *and* Theſaurus's: *Our Taſte and Application are more commenſurate to Manuals and Pamphlets. In Compliance, therefore with the Times, the Editors hereof have choſen the Form and minute Size of a Miſcellany; which, if properly executed, may contain the Uſefulneſs of a Syſtem, without its Dryneſs. They have with Regret obſerved, that ſome Performances of this Kind have already appeared, one of which was ſtifled in its Infancy, by having been nurſed in the Country, where proper Support and Encouragement were not to be had; others yet exiſt in the Land of Literature, but ſo deformed, mutilated, and monſtrous, through the Want of Ability in their Parents and Guardians, that their End is defeated, the Publick receiving no Improvement or Emolument thereby.*

Though this Work is to be miſcellaneous, yet the Deſign of the Editors is to be methodical, without Confuſion and Jumble; though they avoid being ſyſtematical, yet they would be as perſpicuous as poſſible, endeavouring to preſerve a Connexion, as much as the Nature of their Materials will permit, even in the Problems, the moſt anomalous Part, by ſelecting and claſſing them properly.

In the former Part of this Number, after the introductory Diſſertation, is an Extract from Mr. Steele*'s* Conic Sections, *a Treatiſe very ſcarce, but very deſerving, being not to be had in* England. *The ingenious Author has therein demonſtrated the Conic Properties analytically, with great Eaſe and Perſpicuity; though where the Editors have found a more evident Proof, or drawn from more unexceptionable Principles, they have not ſcrupled to inſert it. According to the Reception this meets with, each ſucceeding Number will contain ſome further Extracts till his whole Work be exhauſted.*

The utmoſt Care will be taken, that no Problems be propoſed here, but ſuch as are poſſible, rational, and, as far as may be, applicable to ſome Uſe or Purpoſe in Life; nor ſhall any erroneous Anſwers be inſerted: However, it is hoped, that our Exactneſs in this Particular will not deter our Correſpondents from freely contributing; for the moſt timorous may *be aſſured, nothing ſhall ever appear to his Diſadvantage.*

It is allowed, that whatever is printed and publiſhed, is expoſed to the Animadverſions of all; and every Man is in Duty bound to vindicate Truth, and undeceive the Publick, whenever they find Falſehood obtruded on it, under the ſpecious Appearance of Science; and the rather, becauſe in thoſe Points no Man's Intereſt interferes.

Neither Vanity nor Intereſt are the Motives to this Undertaking, but the pure Love of genuine Truth. To evince this, we have choſen Magna eſt Veritas et prævalebit, *for the Motto of our Frontiſpiece: And if in ſome future Numbers our Zeal for Truth, and Indignation to ſee the Publick impoſed on by Paralogiſms, ſhould prompt us to criticiſe upon ſome late mathematical Performances, wherein, if we, or any of our Correſpondents, ſhould indubitably point out their Errors and Imperfections, and diſcover the true Solutions, we hope the candid World will approve the Attempt; and that thoſe precipitate Authors, who ſo eagerly ruſh on to Fame, will deem it a ſalutary Monition for them (as it will really be intended) to have a more cautious Regard to Truth, by adhering to the true Principles of Science, and preſerving a juſt and logical Reaſoning in their Deductions therefrom. In the*

the Interim, we recommend to their Conſideration raſh Phaeton*'s daring Attempt and dire Diſaſter, and alſo this Adage,* Ne Sutor ultra crepidam.

The EDITORS.

ADVERTISEMENT.

ALL Perſons, who are pleaſed to be Contributors to this Undertaking, by anſwering or ſending new Problems, Paradoxes, Diſſertations, Eſſays, Experiments, or any other Subjects ſuitable to this Deſign, are deſired to direct their Letters, Poſt paid, to Mr. JOHN PELAH, at the *Angel* in *Bunhill-Fields*, near *Moorgate*, *London*, (where the Society meets every *Saturday* Evening) before the Firſt of *January*, 1746: And to prevent Miſtakes, and Ambiguity of Terms, they are alſo deſired to give the Solutions along with the Problems, *&c.* they propoſe.

THE

Mathematician.

A Dissertation *upon the* Origin, Progress, *and* Improvement *of* GEOMETRY.

AMONG thoſe Arguments produced againſt the Opinion of *Ariſtotle*, to prove that the World was not eternal, but had a Beginning; that which is drawn from the late Invention of Arts and Sciences, ſeems to be of great Weight, and almoſt concluſive; for not only theſe, but the neceſſary Affairs of Life, ſuch as Agriculture, Building of Houſes, *&c.* had their Beginning within theſe 4000 Years, or the Compaſs of Hiſtory.

What Soil firſt produced each Science is not quite clear, for the ſame Diſcoveries have appeared in different Parts of the World, without their

their having had any Communication with one another; for Inſtance, the Uſe of the Loadſtone, the Invention of Printing, and Gunpowder, were diſcovered by the *Chineſe* as well as by the *Europeans*.

As to the Methematical Sciences, it ſeems that the Preference therein ſhould be given to the *Europeans*; and 'tis generally allowed that the *Chaldeans* were firſt poſſeſſed of them, eſpecially Aſtronomy, which muſt imply Geometry. Whether *Abraham* taught theſe Sciences firſt to the *Egyptians*, when he went from *Ur* of the *Chaldees*, as ſome learned Men aſſert, is not clear; but on this we may depend, that the *Egyptians* were the firſt People that cultivated Geometry, being compelled thereto by Neceſſity, the Mother of Inventions, in order to aſcertain to every Man his legal Property and Eſtate, in a Country where Boundaries and Land-Marks were ſwept away and confounded by yearly Inundations *.

That the *Egyptians*, in their antient, free, monarchical State, were acquainted with ſome of the ſimple Elements and eaſy Problems in Geometry, is not denied; but we cannot believe they made any great Improvements in the abſtruſe Parts thereof, ſince to *Pythagoras* (the famous Philoſopher of *Samos*, who flouriſhed ſo low as about five hundred and twenty Years before *Chriſt*, and who had lived twenty-two Years in *Egypt)* was attributed the Invention of the thirty-ſecond and forty-ſeventh Propoſitions of the firſt of *Euclid*; for the latter of which he conceived ſo much Joy, that he is ſaid to have offer'd an Hecatomb. A Diſcovery of this Kind, in later Times, would have been entitled but to a ſmall Share of Honour, and the Want of knowing theſe Propoſitions muſt needs make their Geometry very coarſe and imperfect *. Upon this Account, therefore, it may be concluded, that the Learning of the *Egyptians*, for which their Prieſts were ſo famous, and *Moſes* ſo celebrated in holy Writ, for having attained it, did not ſo much conſiſt in Mathematics, as in the Arts of Legiſlation, and civil Polity, and Magic. Their Magicians or wiſe Men thought that the Sun, Moon, Stars, and Elements, were appointed to govern the World; and tho' they acknowledged that God might, upon extraordinary Occaſions, work Miracles, reveal his Will by audible Voices, divine Appearances, Dreams or Prophecies, yet they thought, alſo, that generally ſpeaking, Oracles were given, Prodigies cauſed, Dreams of Things to come occaſioned by the Diſpoſition of the ſeveral Parts of the Univerſe to influence upon one another, at the proper Places and Seaſons, as conſtantly and as neceſſarily as the heavenly Bodies performed their Revolutions; and they imagined that their learned Profeſſors, by a deep Study of, and profound Inquiry into, the Powers of Nature, could make themſelves able to work Wonders, obtain Oracles and Omens, and interpret Dreams, either from Fate, (meaning the natural Courſe of Things) or from Nature, which was when they uſed any artificial Aſſiſtance by Drinks, Inebriations, Diſcipline, or other Means, which were thought to have a natural Power to produce the vaticinal Influence, or prophetic

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* Geometry, like many other Sciences, has outgrown its Name; it originally meant no more than meaſuring the Earth, or ſurveying of Land, as is plain, both from its Etymology, and the principal Uſe that was made of it; whereas now, it means the whole Science of Extenſion and Magnitude, and contemplates the Nature and Properties of all Kinds of Figures, abſtractedly conſidered, without any Regard to Matter.

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* Neither was their Knowledge in Aſtronomy carried to any great Perfection, ſince they were ignorant of the true Length of the Year, taking it to contain only three hundred and ſixty Days, for above a thouſand Years after the Flood.

Frenzy: And in all theſe Particulars they thought the Deity not concerned, but that they were the mere natural Effects of the Influence of the Elements and Planets, at ſet Times and critical Junctures.

From *Egypt* Geometry travelled into *Greece*; for *Thales* the *Mileſian*, who flouriſhed five hundred and eighty-four Years before Chriſt, was the firſt of the *Greeks*, who, coming into *Egypt*, transferred Geometry from thence into *Greece:* He is reputed, certainly, beſides other Things, to have found out the fifth, fifteenth, and twenty-ſixth Propoſitions of *Euclid*'s firſt Book, and the ſecond, third, fourth, and fifth, of the fourth Book. The ſame Perſon improved Aſtronomy, for he began to obſerve the Equinoxes and Solſtices, and was the firſt who foretold an Eclipſe of the Sun.

After him was *Pythagoras*, of *Samos*, before mentioned: This Man much improved and adorned the mathematic Sciences, and ſo attached was he to Arithmetic in particular, that almoſt his whole Method of Philoſophizing was taken from Numbers: He firſt of all abſtracted Geometry from Matter, in which Elevation of Mind he found out ſeveral of *Euclid*'s Propoſitions. He firſt laid open the Matter of incommenſurable Magnitudes, and the five regular Bodies.

Next flouriſhed *Anaxagoras*, of *Clazomenæ*, and *Œnopides*, of *Chios:* Theſe were followed by *Briſo*, *Antipho*, and *Hippocrates*, of *Chios*; which three, for attempting the Quadrature of the Circle, were reprehended by *Ariſtotle*, and, at the ſame Time, celebrated: Then came *Democritus*, *Theodorus*, *Cyrenæus*, and *Plato*, than whom no one brought greater Luſtre to the mathematical Sciences; he amplified Geometry with great and notable Additions, beſtowing incredible Study upon it, and, above all, the Art Analytic, or of Reſolution, was found out by him; the moſt certain Way of Invention and Reaſoning. Upon the Door of his Academy was read this Inſcription, οὐδεὶς ἀγεωμέτρητος εἰσίτω. Thirteen of his familiar Acquaintance are commemorated by *Proclus*, as Men by whoſe Studies the Mathematics were improved. After theſe were *Leon*, and *Eudoxus*, of *Cnidos*, a Man great in Arithmetic, and to whom we owe the whole fifth Book of the Elements; *Xenocrates*, and *Ariſtotle*. To *Ariſteus*, *Iſidore*, and *Hypſicles*, moſt ſubtile Geometricians, we are indebted for the Books of Solids. Afterwards *Euclid* gathered together the Inventions of others, diſpoſed them into Order, improved them, and demonſtrated them more accurately, and left to us thoſe Elements, by which Youth is every where inſtructed in the Mathematics. He died two hundred and eighty-four Years before *Chriſt*. Almoſt an hundred Years after followed *Eratoſthenes*, and *Archimedes*; the Writings of the firſt are loſt, but we have many Remains of the latter. The very Name of *Archimedes* gives an Idea of the Top of human Subtilty, and the Perfection of the whole mathematical Sciences; his wonderful Inventions have been delivered to us by *Polybius*, *Plutarch*, *Tzetres*, and others. He was the firſt who was able to give the exact Quadrature or Menſuration of a Space, bounded by the Arch of a Curve and a right Line, which he did by demonſtrating that the Segment of a Parabola is to its inſcribed Triangle, as 4 : 3. Cotemporary with him was *Conon*; and at no great Diſtance of Time was *Apollonius*, of *Perga*, another Prince in Geometry, called, by Way of Encomium, The great Geometrician. We have extant four Books of Conics in his Name; tho' ſome think *Archimedes* was the Author of them: We have alſo three Books of Spherics by *Theodoſius* the *Tripolite*. In the Year ſeventy, after *Chriſt*, appeared *Claudius Ptolomæus*, the Prince of Aſtronomers, a Man not only moſt ſkilful in Aſtronomy, but in Geometry alſo, as many other Things by him written do witneſs, but eſpecially his Books of Subtenſes. After theſe flouriſhed *Eutocius*, *Cteſibius*, *Proclus*, *Pappus*, and *Theon*. Then enſued a long Period of Ignorance; Arts and Sciences, Liberty and Learning, being driven away and over-run by that brutiſh Herd of Northern Barbarians, whoſe whole Excellence was in their Bones and Muſcles, and Feats of Chivalry their higheſt Ambition. During this diſmal Night of Ignorance, doubtleſs, many curious Diſcoveries, and uſeful Pieces of Knowledge were totally loſt, and the Remainder buried, as it were, in Ruins, till the late Reſtoration of Learning, upon the taking of *Conſtantinople* by the *Turks*, in the Year one thouſand four hundred and fifty-one, after *Chriſt*; whereby the Reſidue of *Greek* and *Roman* Learning was driven, for Refuge, into *Italy*, and the other neighbouring Countries of *Europe*.

Geometry has always heen valued for its extenſive Uſefulneſs, but has been moſt admired for its true and real Excellence, which conſiſts in its Perſpicuity and perfect Evidence: It may, therefore, be of uſe to conſider the Nature of the Demonſtrations, and the Steps by which the Ancients were able, in ſeveral Inſtances, from the Menſuration of right-lined Figures, to judge of ſuch as were bounded by curve Lines; for as they did not allow themſelves to reſolve curvilinear Figures into rectilinear Elements, it is worth While to examine, by what Art they could make a Tranſition from the one to the other.

They found that ſimilar Triangles are to each other in the duplicate Ratio of their homologous Sides; and by reſolving ſimilar Polygons into ſimilar Triangles, the ſame Propoſition was extended to theſe Polygons alſo. But when they came to compare curvilineal Figures, which cannot be reſolved into rectilineal Parts, this Method failed. Circles are the only curvilineal plain Figures conſidered in the Elements of Geometry. If they could have allowed themſelves to have conſidered theſe as ſimilar Polygons of an infinite Number of Sides, (as ſome have ſince done, who pretend to abridge their Demonſtrations) after proving that any ſimilar Polygons inſcribed in Circles, are in the duplicate Ratio of their Diameters, they would have immediately extended this to the Circles themſelves, and would have conſidered 2 *Euc.* 12. as an eaſy Corollary from the firſt: But there is Reaſon to think they would not have admitted a Demonſtration of this Kind, for the old Writers were very careful to admit no precarions Principles, or ought elſe but a few ſelf-evident Truths, and no Demonſtrations but ſuch as were accurately deduced from them. It was a fundamental Principle with them, that the Difference of any two unequal Quantities, by which the greater exceeds the leſſer, may be added to itſelf till it ſhall exceed any propoſed finite Quantity of the ſame Kind: And that they founded their Propoſitions concerning curvilineal Figures upon this Principle. in a particular Manner, is evident from the Demonſtrations, and from the expreſs Declaration of *Archimedes*, who acknowledges it to be a Foundation upon which he eſtabliſhed his own Diſcourſes, and cites it as aſſumed by the Ancients in demonſtrating all the Propoſitions of this Kind: But this Principle ſeems to be inconſiſtent with the admitting of an infinitely little Quantity or Difference, which, added to itſelf any Number of Times, is never ſuppoſed to become equal to any finite Quantity ſoever.

They proceeded, therefore, in another Manner, leſs direct indeed, but perfectly evident. They found that the inſcribed ſimilar Polygons, by having the Number of their Sides increaſed, continually approached to the Areas of the Circles; ſo that the decreaſing Difference between each Circle and its inſcribed Polygon, by ſtill further and further Diviſions of the circular Arches, which the Sides of the Polygon ſubtend, could become leſs than any Quantity that could be aſſigned; and that all this while the ſimilar Polygons obſerved the ſame conſtant invariable Proportions to each other, *viz.* that of the Squares of the Diameters of the Circles. Upon this they founded a Demonſtration, that the Proportion of the Circles themſelves could be no other than that ſame invariable Ratio of the ſimilar inſcribed Polygons. For they proved, by the Doctrine of Proportions only, that the Ratio of the two inſcribed Polygons cannot be the ſame as the Ratio of one of the Circles to a Magnitude leſs than the other, nor the ſame as the Ratio of one of the Circles to a Magnitude greater than the other; therefore the Ratio of the Circles to each other, muſt be the ſame as the invariable Ratio of the ſimilar Polygons inſcribed in them, which is the Duplicate of the Ratio of the Diameters.

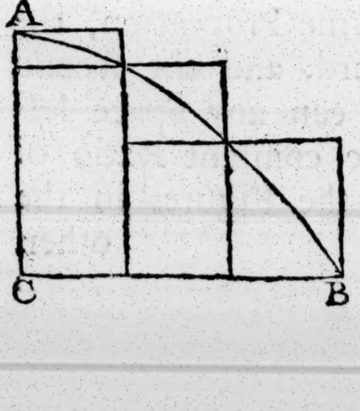
In the ſame Manner the Ancients have demonſtrated, that Pyramids of the ſame Height are to each other as their Baſes, that Spheres are as the Cubes of their Diameters, and that a Cone is the one third Part of a Cylinder on the ſame Baſe, and of the ſame Height. In general, it appears from their Way of Demonſtration, that when two variable Quantities, which always have an invariable Ratio to each other, approach at the ſame Time to two determined Quantities, ſo that they may differ leſs from them than by any aſſignable

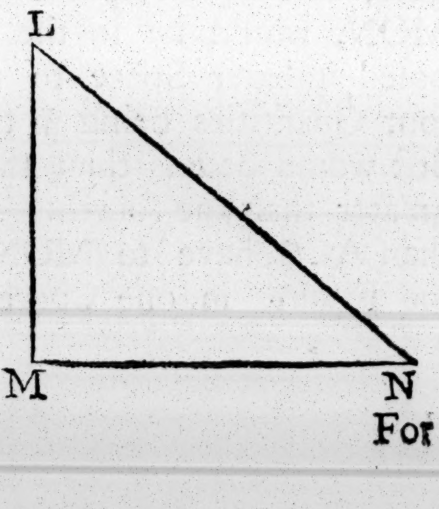
Meaſure; the Ratio of theſe Limits or determined Quantities muſt be the ſame as the invariable Ratio of the two variable Quantities: And this may be conſidered as the moſt ſimple and fundamental Propoſition in this Doctrine, by which we are enabled to compare curvilineal Spaces in ſome of the more ſimple Caſes.

The next Improvement in the Way of demonſtrating among the ancient Geometricians, ſeems to be that which we call the Method of Exhauſtions, which, for the further Illuſtration of this Subject, may be repreſented thus.

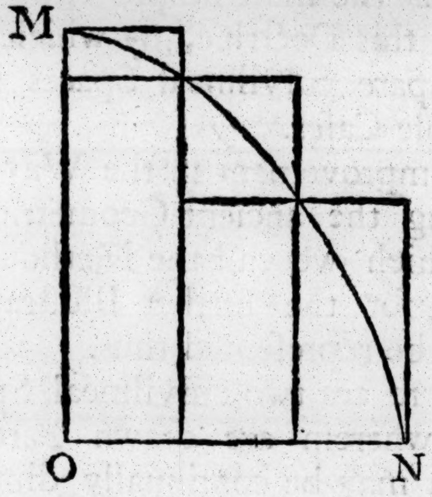
Suppoſe there are two curvilineal Spaces, ACB and MON, wherein are drawn Parallelograms, whoſe Breadth may be continually diminiſhed; it is then obvious, that the firſt circumſcribing, and laſt inſcribing Figures, may be made to differ from that curvilineal Space ACB, and from each other, by leſs than any Space, how minute ſoever, that ſhall be named; *i. e.* the circumſcribed Figure can be made leſs than any Space that exceeds the Curve, and the inſcribed Figure greater than any Space that is leſs than the Curve.

If by conſidering the Properties of theſe inſcribed and circumſcribed Figures, which ariſe from the Nature of the Curve they are adapted to, a right-lined Space LMN may be aſſigned, that ſhall be greater than every inſcribed Figure, and leſs than every circumſcribed Figure, this right-lined Space LMN may be proved to be equal to the curvilineal Space ACB.





For were it greater, a circumſcribed Figure might be made leſs; and if it were leſs, an inſcribed Figure might be made greater.



If, therefore, Parallelograms, whoſe Breadth may be any how diminiſhed, are drawn inſcribing and circumſcribing theſe Curves; and if they are deſcribed in ſuch a Manner, that the circumſcribed Figure of one Curve to the circumſcribed Figure of the other, and the inſcribed to the inſcribed, has one and the ſame conſtant Proportion in every Deſcription: I ſay, that the curve Figure ACB, is to the Curve MON, in the ſame Proportion which the inſcribed and circumſcribed Figures conſtantly bear to each other.

For no Space greater than ACB can have to MON this Ratio, ſince if it could, a Figure might be circumſcribed about ACB leſs than this ſuppoſed greater Space; and this circumſcribed Figure, to the correſponding Figure circumſcribing MON, would be in the ſame Ratio as the ſuppoſed greater Space to the Curve MON; *i. e.* four Quantities being in the ſame Proportion, the firſt would be leſs than the third, and the ſecond greater than the fourth. Nor can any Space leſs than ACB have to MON the conſtant Ratio of the Figures in one Curve to the Figures in the other. For if it could, a Figure might be inſcribed within ACB, which would be greater than this ſuppoſed leſſer Space; and this inſcribed Figure, to its correſpondent Figure inſcribing MON, would be in the ſame Ratio as this imagined leſſer Space to the Curve MON; *i. e.* four Quantities being in the ſame Proportion, the firſt would be greater than the third, and the ſecond leſs than the fourth. Thus no Space but ACB can be to MON in the conſtant Ratio of the circumſcribed and inſcribed Figures.

In the Manner here deſcribed did the antient Geometricians demonſtrate whatever they diſcovered relating to the Dimenſions or Proportions of curve Lines, curvilineal Spaces, and Solids bounded by curve Surfaces; and of which, Sir *Iſaac Newton*'s Doctrine of prime and ultimate Ratios, is no other than an Abbreviation or Improvement in the Form.

Archimedes, indeed, takes a different Way for comparing the Spheroid with the Cone and Cylinder, that is more general, and has a nearer Analogy to the modern Methods. He ſuppoſes the Terms of a Progreſſion to increaſe conſtantly by the ſame Difference, and demonſtrates ſeveral Properties of ſuch a Progreſſion relating to the Sum of the Terms, and the Sum of their Squares; by which he is able to compare the parabolic Conoid, the Spheroid, and hyperbolic Conoid, with the Cone; and the Area of his ſpiral Line with the Area of the Circle. There is an Analogy betwixt what he has ſhewn of theſe Progreſſions, and the Proportions of Figures demonſtrated in the Elementary Geometry; the Conſequence of which may illuſtrate his Doctrine, and ſerve, perhaps, to ſhew that it is more regular and compleat in its Kind than ſome have imagined. The Relation of the Sum of the Terms to the Quantity that ariſes by taking the greateſt of them as often as there are Terms, is illuſtrated by comparing the Triangle with a Parallelogram of the ſame Height and Baſe; and what he has demonſtrated of the Sum of the Squares of the Terms compared with the Square of the greateſt Term, may be illuſtrated by the Proportion of the Pyramid to the Priſm, or of the Cone to the Cylinder, their Baſes and Heights being equal; and by the Ratios of certain Fruſtums or Proportions of theſe Solids deduced from the elementary Proportions.

He appears ſolicitous, that his Demonſtrations ſhould be found to depend on thoſe Principles only, that had been univerſally received before his Time. In his Treatiſe of the Quadrature of the Parabola, he mentions a Progreſſion, whoſe Terms decreaſe conſtantly in the Proportion of four to one; but he does not ſuppoſe this Progreſſion to be continued to Infinity, or mention the Sum of an infinite Number of Terms; tho' it is manifeſt, that all which can be underſtood by thoſe who aſſign that Sum, was fully known to him. He appears to have been more fond of preſerving to the Science all its Accuracy and Evidence, than of advancing Paradoxes; and contents himſelf with demonſtrating this plain Property of ſuch a Progreſſion, That the Sum of the Terms continued at Pleaſure, added to the third Part of the laſt Term, amounts always to $\frac{4}{3}$ of the firſt Term: Nor does he ſuppoſe the Chords of the Curve to be biſected to Infinity; ſo that after an infinite Biſection, the inſcribed Polygon might be ſaid to coincide with the Parabola. Theſe Suppoſitions had been new to the Geometricians in his Time, and ſuch he appears to have carefully avoided.

This is a ſummary Account of the Progreſs that was made by the Ancients in meaſuring and comparing curvilineal Figures, and of the Method by which they demonſtrated their Theorems of this Kind. It is often ſaid, that curve Lines have been conſidered by them as Polygons of an infinite Number of Sides; but this Principle no where appears in their Writings: We never find them reſolving any Figure or Solid into infinitely ſmall Elements: On the contrary, they ſeem to have avoided ſuch Suppoſitions, as if they judged them unfit to be received into Geometry, when it was obvious, that their Demonſtrations might have been ſometimes abridged by admitting them. They conſidered curvilineal Areas as the Limits of circumſcribed or inſcribed Figures, of a more ſimple Kind, which approach to theſe Limits (by a Biſection of Lines or Angles, continued at Pleaſure) ſo that the Difference between them may become leſs than any given Quantity. The incribed or circumſcribed Figures were always conceived to be of a Magnitude and Number that is aſſignable; and from what had been ſhewn of theſe Figures, they demonſtrated the Menſuration, or the Proportions of the curvilineal Limits themſelves, by Arguments *ab Abſurdo.* They had made frequent Uſe of Demonſtrations of this Kind from the Beginning of the Elements; and theſe are, in a particular Manner, adapted for making a Tranſition from right-lined Figures, to ſuch as are bounded by curve Lines. By admitting them only, they eſtabliſhed the more difficult and ſublime Part of their Geometry, on the ſame Foundation as the firſt Elements of the Science; nor could they have propoſed to themſelves a more perfect Model.

But as theſe Demonſtrations, by determining diſtinctly all the ſeveral Magnitudes and Proportions of theſe inſcribed and circumſcribed Figures, did frequently extend to very great Lengths, other Methods of demonſtrating have been contrived by the Moderns, whereby to avoid theſe circumſtantial Deductions. The firſt Attempt of this Kind known to us, was made by *Lucas Valerius*; but afterwards *Cavalerius*, an *Italian*, about the Year one thouſand ſix hundred and thirty five, advanced his Method of Indiviſibles, in which he propoſes, not only to abbreviate the antient Demonſtrations, but to remove the indirect Form of Reaſoning uſed by them, of proving the Equality or Proportion between Lines and Spaces, from the Impoſſibility of their having any different Relation; and to apply to theſe curved Magnitudes the ſame direct Kind of Proof that was before applied to right-lined Quantities.

This Method of comparing Magnitudes, invented by *Cavalerius*, ſuppoſes Lines to be compounded of Points, Surfaces of Lines, and Solids of Planes; or, to make uſe of his own Deſcription, Surfaces are conſidered as Cloth, conſiſting of parallel Threads; and Solids are conſidered as formed of parallel Planes, as a Book is compoſed of its Leaves, with this Reſtriction, that the Threads or Lines, of which Surfaces are compounded, are not to be of any conceivable Breadth, nor the Leaves or Planes of Solids of any Thickneſs. He then forms theſe Propoſitions, that Surfaces are to each other, as all the Lines in one to all the Lines in the other; and Solids, in like Manner, in the Proportion of all the Planes.

This Method exceedingly ſhortened the former tedious Demonſtrations, and was eaſily perceived; ſo that Problems, which at firſt Sight appeared of an inſuperable Difficulty, were afterwards reſolved, and came, at length, to be deſpiſed, as too ſimple and eaſy: The Menſuration of Parabolas, Hyperbolas, Spirals of all the higher Orders, and the famous Cycloid, were among the early Productions of this Period. The Diſcoveries made by *Torricelli*, *de Fermat*, *de Roberval*, *Gregory*, *St. Vincent*, &c. are well known. They who have not read many Authors, may find a Synopſis of this Method in *Ward's Young Mathematician's Guide*, where he treats of the Menſuration of Superficies and Solids.

Notwithſtanding, as this Method is here explained, it is manifeſtly founded on inconſiſtent and impoſſible Suppoſitions; for while the Lines, of which Surfaces are ſuppoſed to be made up, are real Lines of no Breadth, it is obvious, that no Number, whatever, of them, can form the leaſt imaginable Surface: If they are ſuppoſed to be of ſome ſenſible Breadth, in order to be capable of filling up Spaces, *i. e.* in Reality to be Parallelograms, how minute ſoever be their Altitude, the Surfaces may not be to each other in the Proportion of all ſuch Lines in one, to all the like Lines in the other; for Surfaces are not always in the ſame Proportion to each other with the Parallelograms inſcribing them.

The ſame contradictory Suppoſitions do obviouſly attend the Compoſition of Solids by parallel Planes, or of Lines by ſuch imaginary Points.

This heterogeneous Compoſition of Quantity, and Confuſion of its Species, ſo different from that Diſtinctneſs, for which the Mathematics were ever famous, was oppoſed at its firſt Appearance by ſeveral eminent Geometricians, particularly by *Guldinus* and *Tacquet*, who not only excepted to the firſt Principles of this Method, but taxed the Concluſions formed upon it as erroneous. But as *Cavalerius* took Care, that the Threads or Lines of which the Surfaces to be compared together were formed, ſhould have the ſame Breadth in each (as he himſelf expreſſes it) the Concluſions deduced by his Method, might generally be verified by ſounder Geometry; ſince the Compariſon of theſe

Lines was, in Effect, the comparing together the inſcribed Figures.

As in the Application of this Method, Error, by proper Caution, might be avoided, the Aſſiſtance it ſeemed to promiſe in the analytical Part of Geometry, made it eagerly followed by thoſe who were more deſirous to diſcover new Propoſitions, than ſolicitous about the Elegance or Propriety of their Demonſtrations. Yet ſo ſtrange did the contradictory Conception appear, of compoſing Surfaces out of Lines, and Solids out of Planes, that, in a ſhort Time, it was new modelled into that Form, which it ſtill retains, and which now univerſally prevails among the foreign Mathematicians, under the Name of the differential Method, or the Analyſis of infinitely Littles.

In this reformed Notion of Indiviſibles, Surfaces are now ſuppoſed as compoſed not of Lines, but of Parallelograms, having infinitely little Breadths and Solids, in like Manner as found of Priſms, having infinitely little Altitudes. By this Alteration it was imagined, that the heterogeneous Compoſition of *Cavalerius* was ſufficiently evaded, and all the Advantages of his Method retained. But here, again, the ſame Abſurdity occurs as before; for if by the infinitely little Breadth of theſe Parellelograms, we are to underſtand what theſe Words literally import, *i. e.* no Breadth at all; then they cannot, any more than the Lines of *Cavalerius*, compoſe a Surface; and if they have any Breadth, the right Lines bounding them cannot coincide with a Surface bounded by a curve Line.

The Followers of this new Method grew bolder than the Diſciples of *Cavalerius*, and having tranſformed his Points, Lines, and Planes, into infinitely little Lines, Surfaces, and Solids, they pretended, they no longer compared together heterogeneous Quantities, and inſiſted on their Principles, being now become genuine; but the Miſtakes they frequently fell into, were a ſufficient Confutation of their Boaſts; for, notwithſtanding this new Model, the ſame Limitations and Cautions were ſtill neceſſary: For Inſtance, this Agreement between the inſcribing Figures and the curved Spaces to which they are adapted, is only partial; and in applying their Principles to Propoſitions already determined by a juſter Method of Reaſoning, they eaſily perceived this Defect; both in Surfaces and Solids, it was evident, at firſt View, that the Perimeters diſagreed. And as no one Inſtance can be given, where theſe indiviſible or infinitely little Parts do ſo compleatly coincide with the Quantities they are ſuppoſed to compound, as in every Circumſtance to be taken for them, without producing erroneous Concluſions, ſo we find, where a ſurer Guide was wanting, or diſregarded, theſe Figures were often imagined to agree, where they ought to have been ſuppoſed to differ.

Leibnitz, in two Diſſertations, one on the Reſiſtance of Fluids, the other on the Motion of the heavenly Bodies, has, on this Principle, reaſoned falſly concerning the Lines intercepted between Curves and their Tangents. *Berneulli* has, likewiſe, made the ſame Miſtake in a Diſſertation on the Reſiſtance of Fluids, and in a pretended Solution of the Problem concerning iſoperimetrical Curves. Nay, Mr. *Parent* has had the Raſhneſs to oppoſe erroneous Deductions from this abſurd Principle, to the moſt indubitable Demonſtrations of the great *Huygens*. Thus it appears, that the Doctrine of Indiviſibles contains an erroneous Method of Reaſoning, and, in Conſequence thereof, in every new Subject to which it ſhall be applied, is liable to freſh Errors.

It is alſo manifeſt, that the great Brevity it gave to Demonſtrations, aroſe entirely from the abſurd Attempt of comparing curvilineal Spaces in the ſame direct Manner as right-line Figures can be compared; for, in order to conclude directly the Equality or Proportion of ſuch Spaces, no Scruple was made of ſuppoſing, contrary to Truth, that rectilineal Figures, capable of ſuch direct Compariſon, could adequately fill up the Spaces in Queſtion; whereas, the Doctrine of Exhauſtions does not attempt, from the Equality or Proportion of the inſcribing or circumſcribing Figures, to conclude, directly, the like Proportions of theſe Spaces, becauſe thoſe Figures can never, in Reality, be made equal to the Spaces they are adapted to: But as theſe Figures may be made to differ from the Spaces to which they are adapted, by leſs than any Space propoſed, how minute ſoever, it ſhews, by a juſt, tho' indirect Deduction from theſe circumſcribing and inſcribing Figures, that the Spaces whoſe Equality is to be proved, can have no Difference; and that the Spaces, whoſe Proportion is to be ſhewn, cannot have a different Proportion than that aſſigned them.

The *Arithmetica Infinitorum* of Dr. *Wallis*, was the fulleſt Treatiſe of this Kind that appeared before the Invention of Fluxions. *Archimedes* had conſidered the Sums of the Terms in an arithmetical Progreſſion, and of their Squares only, (or rather the Limits of theſe Sums only) theſe being ſufficient for the Menſuration of the Figures he had examined. Dr. *Wallis* treats this Subject in a very general Manner, and aſſigns like Limits for the Sums of any Powers of the Terms, whether the Exponents be Integers or Fractions, poſitive or negative. Having diſcovered one general Theorem that includes all of this Kind, he then compoſed new Progreſſions from various Aggregates of theſe Terms, and enquired into the Sums of the Powers of theſe Terms, by which he was enabled to meaſure accurately, or by Approximation, the Areas of Figures without Number: But he compoſed this Treatiſe (as he tells us) before he had examined the Writings of *Archimedes*; and he propoſes his Theorems and Demonſtrations in a leſs accurate Form: He ſuppoſes the Progreſſions to be continued to Infinity, and inveſtigates, by a Kind of Induction, the Proportion of the Sum of the Powers, to the Production that would ariſe by taking the greateſt Power as often as there are Terms. His Demonſtrations, and ſome of his Expreſſions, (as when he ſpeaks of Quantities more than infinite) have been excepted againſt; however, it muſt be owned, this valuable Treatiſe contributed to produce the great Improvements which ſoon after followed.

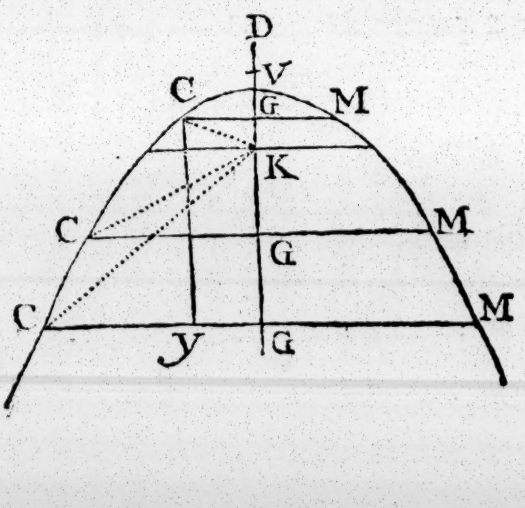
But Sir *Iſaac Newton* has accompliſhed what *Cavalerius* wiſhed for, by inventing the Method of Fluxions, and propoſing it in a Way that admits of ſtrict Demonſtration, which requires the Suppoſition of no Quantities, but ſuch as are finite, and eaſily conceived; by his Doctrine of prime and ultimate Ratios, he has found out the proper Medium, whereby to avoid the impoſſible Notion of Indiviſibles on the one hand, and the Length of Exhauſtions on the other. The Computations in this Method, are the ſame as in the Method of Infiniteſimals, but it is founded on accurate Principles, agreeable to the antient Geometry; in it the Premiſes and Concluſions are equally accurate, no Quantities are rejected as infinitely ſmall, and no Part of a Curve is ſuppoſed to coincide with a right Line: But as the Explication of their Nature and Uſe has employed ſome of our greateſt Mathematicians to write expreſs Treatiſes thereon; and as the Invention can never be ſufficiently applauded, we will conclude with Mr. *Ditton*, that the next Improvement muſt be the Science of pure Intelligences.

CONIC SECTIONS.

Of the PARABOLA.

The GENESIS.

IF from a Point V, in any indefinite right Line, there be taken VD=VK, and from the Point K, as a Center, with the Diſtance DG, you interſect CM drawn perpendicular to DG, thoſe Points will be in the Curve of a Parabola; and proceeding in this Manner, an indefinite Number of Points may be found, thro' which, if a Line be ſuppoſed drawn, the Space (CVM) comprehended thereby, and any right Line drawn at right Angles, to the above indefinite Line, will be a Parabola.



DEFINITIONS.

1. The Point V is the Vertex, and K the Focus of the Parabola.

2. The right Line DG, paſſing thro' the Focus, is called the Axis.

3. A right Line drawn perpendicular to the Axis, terminating in the Curve, is called an Ordinate to the Axis, as GC.

4. The Diſtance, in the Axis, from the Vertex to the Interſection of the Ordinate, is called the Abſciſſa, correſponding to that Ordinate, as VG.

5. A right Line drawn from any Point in the Curve, and parallel to the Axis, is called a Diameter, as CY; and the Point in the Curve, from whence it is drawn, the Vertex of that Diameter.

PROPOSITION I.

The Square of any Ordinate, is equal to the Rectangle of the Abſciſſa of that Ordinate, drawn into quadruple the Diſtance of the Focus from the Vertex; that is, $\overline{GC}^2 = VG \times 4KV$.

DEMONSTRATION.

Let $KV = VD = q$, $VG = x$, and $GC = y$; then, by the Geneſis, $GK = q \backsim x$, and $DG = CK = q + x$; but (by *Eu.* 1. 47.) $\overline{KC}^2 = \overline{KG}^2 + \overline{GC}^2$; that is, $q^2 + 2qx + x^2 = q^2 - 2qx + x^2 + y^2$, or $4qx = y^2$; that is, $\overline{GC}^2 = VG \times 4KV$. Q.E.D.

COROLLARY I.

The Squares of the Ordinates, are to each other, as their reſpective Abſciſſas; becauſe $Y^2 = 4qX$, and $y^2 = 4qx$; therefore, $Y^2 : y^2 :: 4qX : 4qx :: X : x$.

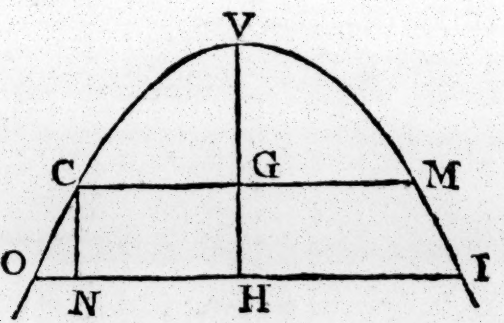
Definition. Quadruple the Diſtance of the Focus from the Vertex, is called the Parameter of the Axis, and is a third Proportional to any Abſciſſa, and its correſponding Ordinate. For by putting $4q=p$, we have $px=y^2$; therefore, $x:y::y:p$.

COROLLARY II.

The Ordinate paſſing thro' the Focus, is equal to Half the Parameter of the Axis. For in this Caſe $x=q$; therefore (by the Propoſition) $4q^2=y^2$; whence $y=2q=\frac{1}{2}p$.

PROPOSITION II.

As the Parameter of the Axis, is to the Sum of any two Ordinates, ſo is their Difference, to the Difference of their Abſciſſas; that is, $p:IN::NO:NC$.

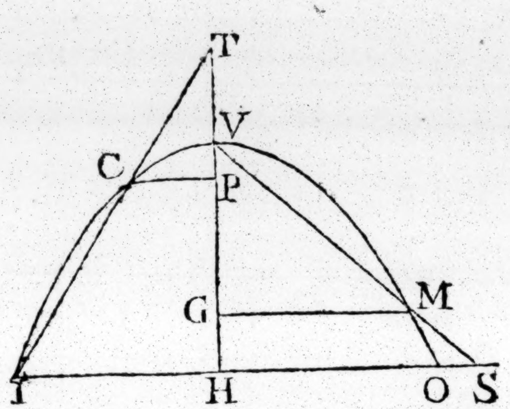


DEMONSTRATION.

Let $HO=Y$, $GC=y$, $VH=X$, and $VG=x$; then (by Prop. 1.) $pX=Y^2$, and $px=y^2$; therefore, $pX-px=Y^2-y^2$; conſequently, $p:Y+y::Y-y:X-x$; that is, $p:IN::NO:NC$. Q. E. D.

PROPOSITION III.

If from the Vertex a right Line be drawn, ſo as to cut the Curve, and continued till it meet any Ordinate produced, it will be, as the Parameter of the Axis, is to the Ordinate drawn from the Interſection with the Curve, ſo is the produced Ordinate, to its Abſciſſa; that is, $p:\mathrm{GM}::\mathrm{HS}:\mathrm{HV}$.



DEMONSTRATION.

Let $\mathrm{HS}=b$, $\mathrm{VG}=x$, $\mathrm{VH}=\mathrm{X}$, $\mathrm{HO}=\mathrm{Y}$, and $\mathrm{GM}=y$; then (by Prop. 1.) $y^2:\mathrm{Y}^2::x:\mathrm{X}::$ by ſimilar Triangles, $y:b$; therefore, $\frac{by}{y}=\mathrm{Y}^2=p\mathrm{X}$; or $by=p\mathrm{X}$; that is, $p:\mathrm{GM}::\mathrm{HS}:\mathrm{HV}$. Q.E.D.

PROPOSITION IV.

If from any Point D, in the Axe produced, a right Line be drawn interſecting the Curve in two Points C and I, and the Ordinates CP, IH, be drawn from the ſaid Interſections; VD will be a mean Proportional between VP and VH.

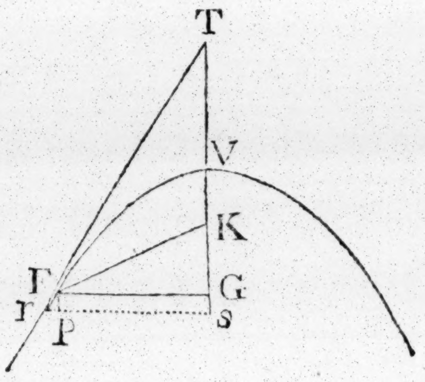
DEMON-

DEMONSTRATION.

Let $VD = b$, $VP = x$, $VH = X$; then $PD = b + x$, and $HD = b + X$; but from ſimilar Triangles $\overline{PD}^2 : \overline{HD}^2 :: \overline{PC}^2 : \overline{IH}^2 ::$ (by Prop. 1.) $PV : HV$; that is, $b^2 + 2bx + x^2 : b^2 + 2bX + X^2 :: x : X$; therefore $\overline{X - x} \times b^2 = \overline{X - x} \times xX$; or $b^2 = xX$; that is, $x : b :: b : X$; or $VP : VD :: VD : HV$. Q. E. D.

PROPOSITION V.

If any right Line touch the Curve, and an Ordinate be drawn from the Point of Contact; then the Abſciſſa correſponding that Ordinate, will be equal to the Diſtance, in the Axe produced, from the Vertex of the Curve, to the Interſection of the Tangent; that is, $GV = VT$.



DEMONSTRATION.

Let rF be an indefinitely ſmall Part of the Curve, and continued to T; draw rs parallel to the Ordinate, and Fp parallel to the Axe, and let $Fp = n$, $rp = m$, $VT = a$, and the other Symbols remaining as uſual; then $VS = n + x$, and $rs = m + y$; whence, by ſimilar Triangles, $m : n :: y : a + x$; therefore $\frac{ny}{m} = a + x$; but (by Prop. 1.) $p \times VS = \overline{Sr}^2$, and $p \times VG = \overline{GF}^2$; that is, $pn + px = m^2 + 2my + y^2$, and $px = y^2$; therefore $y^2 + 2my - pn = px = y^2$; that is, $n = \frac{2my}{p}$, and conſequently $x + a = \frac{2my}{p} \times \frac{y}{m} = \frac{2y^2}{p} = \frac{2px}{p} = 2x$; therefore $a = x$, or $VT = GV$. Q. E. D.

PROPOSITION VI.

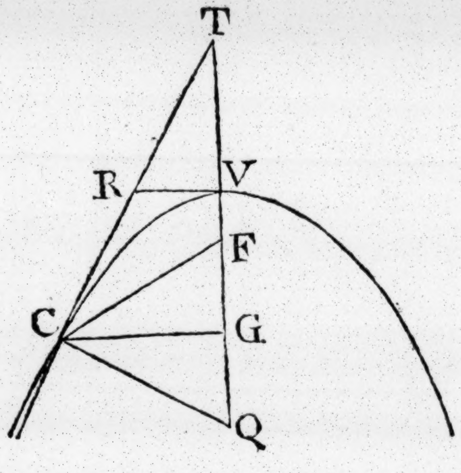
If from the Point of Contact, a right Line be drawn to the Focus, it will be equal to the Diſtance, in the Axe produced, from the Focus to the Interſection of the Tangent; that is, $KF = KT$.

DEMONSTRATION.

By the laſt (Propoſition) $GT = 2x$, and by the firſt, $KG = x - \frac{1}{4}p$; conſequently $KT = GT - KG = 2x - x + \frac{1}{4}p = x + \frac{1}{4}p$. = by the Geneſis to KF. Q. E. D.

PROPOSITION VII.

If to the Tangent, from the Point of the Contact, a Perpendicular be drawn, and produced to meet the Axe; then the Diſtance in the Axe from that Point, to the Ordinate drawn from the Point of Contact, that is, the *Subnormal*, is equal to half the Parameter of the Axe; that is, QG $= \frac{1}{2} p$.



DEMONSTRATION.

Let QG $= b$; then (by *Eu.* 8. 6.) GT : GC :: GC : GQ; that is, $2x : y :: y : b$; therefore, $2bx = y^2 = px$; whence $b = \frac{1}{2} p$, or QG $= \frac{1}{2} p$. Q.E.D.

PROPOSITION VIII.

The Diſtance from the Focus to the Point of Contact; from the Focus to the Interſection of the Tangent with the Axe, and from the Focus to the End of the Subnormal, are equal to each other; that is, FC = FT = FQ.

DEMONSTRATION.

From the Geneſis GF $= x - \frac{1}{4} p$, and (by Prop. 7th) QG $= \frac{1}{2} p$; therefore FG = GF + GQ $= x + \frac{1}{4} p =$ (by the Geneſis) FC = (by Prop. 6th) FT. Q.E.D.

COROLLARY I.

Hence F is the Center of a Circle, the Periphery of which will paſs through the Points denoted by Q, C, and T.

COROLLARY II.

The Angle formed by the Tangent and Axe, is equal to half the Angle formed by the Axe, and a right Line drawn from the Point of Contact; that is, the Angle $CTG = \frac{1}{2}$ the Angle CFG, by *Eu.* 31. 3.

PROPOSITION IX.

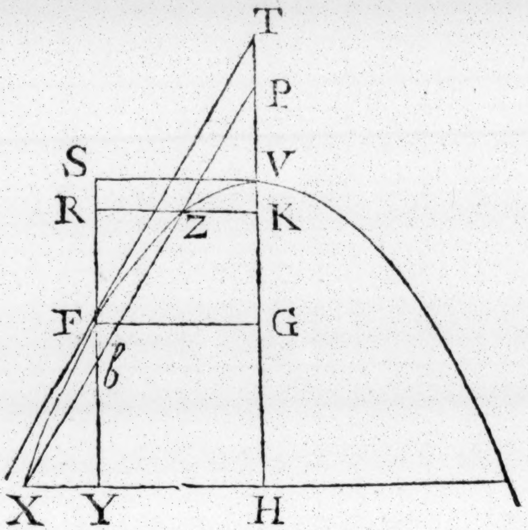
If from the Vertex, a right Line be drawn parallel to an Ordinate, drawn from the Point of Contact, to meet the Tangent; the Square of that Line, will be equal to the Rectangle of half the Parameter of the Axis, drawn into half the Abſciſſa correſponding that Ordinate; that is, $\overline{VR} = \frac{1}{2}p \times \frac{1}{2}GV$.

DEMONSTRATION.

Let $VR = b$; then, becauſe the Triangles TVR, TGC are ſimilar, and $VT = \frac{1}{2}GT$ (by Prop. 5;) therefore, $VR = \frac{1}{2}GC$; that is, $b = \frac{1}{2}y$; whence $b^2 = \frac{1}{4}y^2 =$ (by Prop. 1.) $\frac{1}{2}p \times \frac{1}{2}x$, or $\overline{VR}^2 = \frac{1}{2}p \times \frac{1}{2}GV$. Q. E. D.

PROPOSITION X.

If to the Tangent drawn to the Vertex of any Diameter, a right Line be drawn parallel; the Part of that Line which lies within the Curve, will be biſſected by that Diameter; that is, the Ordinate $Xb=bZ$.



DEMONSTRATION.

Draw KR, VS parallel to the Ordinate FG, meeting the Diameter FY produced in R and S; through F, where that Diameter interſects the Curve, draw the Tangent FT, and through Z, where KR interſects the Curve, draw PX parallel to that Tangent, meeting the Curve in X; then,

1. Becauſe GT is double VG (by Prop. 5.) the Triangle $FGT = \square SG$, and ſince PX is parallel to TF, the Triangles KZP, FGT are ſimilar; wherefore $\triangle GFT$, or $\square GS : \triangle KZP :: \overline{GF}^2 : \overline{KZ}^2 ::$ (by Cor. 1. Prop. 1.) $GV : KV ::$ (by *Eu.* 1. 6.) $\square GS : \square KS$; therefore $\triangle KZP = \square KS$.

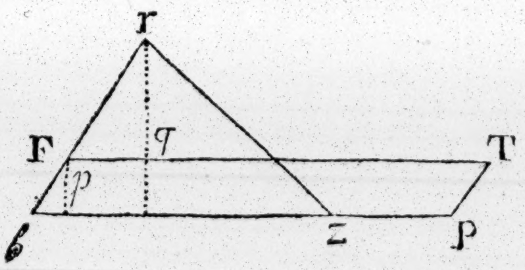
2. The Triangle GFT, or $\square GS : \triangle HXP :: \overline{GF}^2 : \overline{HX}^2 ::$ (by Cor. 1. Prop. 1) $GV : HV :: \square GS : \square HS$; therefore $\triangle HXP = \square HS$; from which taking the common Figure $HYbZK$, there remains $\triangle YXb =$ and ſimilar to the Triangle bRZ, and conſequently $Xb = bZ$. Q. E. D.

COROLLARY.

The Figure $bFTP = \triangle YXb$, becauſe the Triangle $GFT = \square HS$; therefore the Figure $HYFT = \square HF + \triangle GFT = \square HF + \square GS = \square HS = \triangle HXP$; from which taking the common Figure $HYbP$, there remains the Figure $bFTP = \triangle YXb$.

LEMMA.

If FT be parallel to bp, and the Triangle brz, equal to the Trapezium $FbTp$; then will $\overline{FT + bp} \times bF = br \times bz$.



DEMONSTRATION.

Becauſe, by Hypotheſis, $\overline{FT+bp}\times p = zb\times q$; therefore, $FT+bp : zb :: q : p ::$ (by ſimilar Triangles) $rb : bF$; conſequently $\overline{FT+bp}\times bF = br\times bz$. Q. E. D.

Definition. Let $FS : FO :: 2FT : P$, the Parameter belonging to the Diameter FY; then,

PROPOSITION XI.

The Parameter ſo obtained, drawn into an Abſciſſa of that Diameter, is equal to the Square of the Ordinate correſponding that Abſciſſa; that is, $P\times bF = \overline{bX}^2 = \overline{bZ}^2$. (See Fig. preceding the Lemma.)

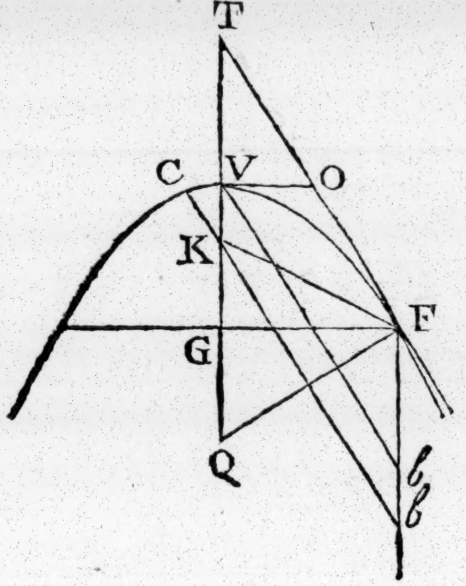
DEMONSTRATION.

Since, by the Definition, $\frac{P}{2FT} = \frac{FO}{FS} =$ by ſimilar Triangles $\frac{bX}{bY}$; and (by the preceding Lemma) $2FT\times bF = bY\times bX$; therefore $\frac{P}{2FT}\times 2FT\times bF = \frac{bX}{bY}\times bY\times bX$; or $P\times bF = \overline{bX}^2 = \overline{bZ}^2$. Q. E. D.

PROPOSITION XII.

The Parameter of any Diameter, is equal to the Parameter of the Axis, added to quadruple the

Abſciſſa of the Ordinate, drawn from the Vertex of that Diameter; that is, $P = p + 4GV$.



From the Vertex draw Vp parallel to the Tangent FT, which (by Prop. 10.) will be an Ordinate to the Diameter FY; then by reaſon of Parallels $bF = VT =$ (by Prop. 5.) $GV = x$; and by the laſt $Px = \overline{bV}^2 = \overline{FT}^2 = \overline{FG}^2 + \overline{GT}^2 = 4x^2 + y^2 = 4x^2 + px$; conſequently $P = 4x + p = p + 4GV$. Q. E. D.

PROPOSITION XIII.

The Diſtance from the Focus, to the Vertex of any Diameter, is equal to one Fourth of the Parameter of that Diameter; that is, $KF = \frac{1}{4}P$.

DEMONSTRATION.

By the laſt Propoſition, $P = p + 4VT$, and by the firſt, $p = 4KV$; therefore $P = 4KV + 4VT$, and conſequently $\frac{1}{4}P = KV + VT = KT =$ (by Prop. 6.) KF. Q. E. D.

PROPOSITION XIV.

If from the Focus, a Perpendicular be drawn to any Tangent; then the Square of that Line, will be equal to the Rectangle under the Focal-diſtance, and the Diſtance of the Point of Contact from the Focus; that is, $\overline{KO}^2 = KV \times KF$.

DEMONSTRATION.

From the Vertex V, draw VO parallel to GF, which will coincide with the Point O; becauſe (by Prop. 5.) $GV = VT$; therefore (by *Eu.* 2. 6.) $TO = OF$; alſo becauſe the Angle KOT is right (by *Eu.* 8. 6.) $TK : KO :: KO : KV$; conſequently $\overline{KO}^2 = KV \times TK = KV \times KF$. Q. E. D.

PROPOSITION XV.

If an Ordinate to any Diameter paſs thro' the Focus; then the Abſciſſa correſponding to that Ordinate, will be equal to one Fourth, and the Ordinate equal to one Half of the Parameter correſponding to that Diameter.

DEMONSTRATION.

Becauſe bC is parallel to FT, and bF parallel to KT; therefore $bF = KT =$ (by Prop. 6.) $KF =$ (by Prop. 13.) $\frac{1}{4}P$.

2. Since $bF = \frac{1}{4}P$, and (by Prop. 11.) $P \times bF = \overline{bC}^2$; therefore $\frac{1}{4}P^2 = \overline{bC}^2$, and conſequently $\frac{1}{2}P = bC$. Q. E. D.

PROPOSITION XVI.

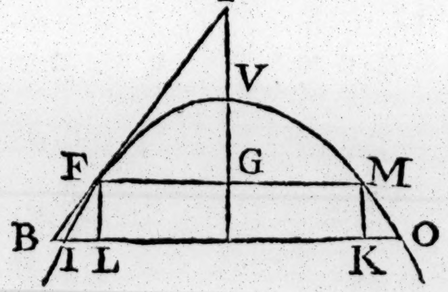
The Diſtance in the Axis, from the Interſection of the Tangent, to the End of the Subnormal, is equal to half the Parameter of that Diameter, whoſe Vertex is the Point of Contact; that is $QT = \frac{1}{2}P$.

DEMONSTRATION.

Since (by Prop. 13) $FK = \frac{1}{4}P$, and (by Prop. 8.) $FK = QK = KT$; conſequently $QT = QK + KT = \frac{1}{4}P + \frac{1}{4}P = \frac{1}{2}P$. Q. E. D.

PROPOSITION XVII.

If a double Ordinate be drawn from the Point of Contact, and another double Ordinate be drawn below to meet the Tangent produced; then as the double Ordinate paſſing through the Point of Contact, is to the Sum of the two Ordinates, ſo is their Difference, to the external Part of the lower Ordinate added to the Difference of the Ordinates; that is, MF : OL :: IL : BL.



DEMON-

DEMONSTRATION.

Let $VG = x$, then (by Prop. 5.) $GT = 2x$, $FG = y$, $OL = c$, $IL = m$, and $LB = d$; then (by Prop. 2.) $p : c :: m : \frac{cm}{p} = LF$, and by ſimilar Triangles, $2x : y :: \frac{mc}{p} : d$; therefore $2pdx = mcy$; but becauſe $px = y^2$; (by Prop. 1.) $2dy = mc$, or $2y : c :: m : d$; that is, $MF : OL :: IL : BL$. Q. E. D.

PROPOSITION XVIII.

The ſame Things being ſuppoſed as before; the Difference of the Ordinates, is a Mean-proportional, between the Double of the upper Ordinate, and the external Part of the lower; that is, $FM : IL :: IL : BI$.

DEMONSTRATION.

Let $BI = c$, $IL = m$, and $FM = 2y$; then $OL = m + 2y$, and (by Prop. 17.) $2y : m + 2y :: m : d$; therefore $d = \frac{m^2 + 2my}{2y}$, and conſequently $c = d - m = \frac{m^2 + 2my}{2y} - m = \frac{m^2}{2y}$; that is, $FM : IL :: IL : BL$. Q. E. D.

PROPOSITION XIX.

The ſame Things being ſtill ſuppoſed; as the Double of the lower Ordinate added to the external Part, is to the Sum of the two Ordinates, ſo is the external Part of the lower Ordinate added to the Difference of the Ordinates, to the Difference of the Ordinates; that is, OB : LB ∷ OL : IL.

DEMONSTRATION.

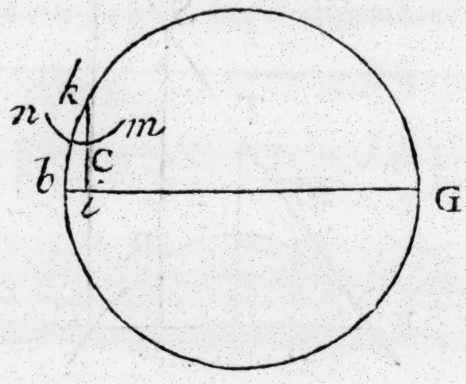
Let OL $= c$, LB $= d$, IL $= m$; then OB $= c + d$, and MF $= c - m$; but (by Prop. 17.) MF : OL ∷ IL : BL; that is, $c - m : c :: m : d$; therefore $cd - dm = cm$, and conſequently $cm + dm = cd$; or $c + d : c :: d : m$; that is, OB : LB ∷ OL : IL. Q. E. D.

PROPOSITION XX.

Still ſuppoſing the ſame Things; having OI, and BI given, it is propoſed to find IL.

SOLUTION.

Let KL $= b$, IL $= m$, BI $= a$, and OI $= c$; then (by Prop. 18) KL : (MF) IL ∷ IL : BI; that is, $b : m :: m : a$; therefore $ba = m^2$ and $b = \frac{m^2}{a}$; alſo $c = b + 2m = \frac{m^2}{a} + 2m = \frac{m^2 + 2am}{a}$; therefore $ac = m^2 + 2am$, and, conſequently, by Reduction, $m = \sqrt{\overline{a+c} \times a} - a$, Q. E. I.

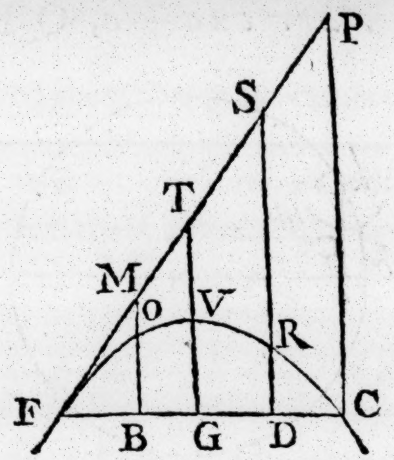


COROLLARY.

Hence from a Point B, without the Curve, and not in the Axe produced, may be drawn a Tangent. For if from the given Point B, the right Line BO be drawn perpendicular to the Axe, meeting the Curve in O; let *b* G be taken equal to 2 BI + OI, about which as a Diameter, let a Circle be deſcribed; in that Diameter, take *b i* equal to BI, and from the Point *i*, erect the Perpendicular *i k*, meeting the Periphery in *k*; from *k* as a Center, with the Interval BI, deſcribe *n c m* cutting *i k* in C; laſtly, in IO, take IL equal to *c i*, and draw LF parallel to the Axe, and the Point F will be determined; through which if a right Line be drawn from the given Point B, terminating in the Axe produced, it will be a Tangent to the Curve.

PROPOSITION XXI.

If FP touch the Curve in F, and from any two Points M, S in that Tangent, the right Lines BM, SD be drawn parallel to the Axe, meeting the Ordinate in B and D; then will MO : SR :: $\overline{FB}^2$: $\overline{FD}^2$.



DEMONSTRATION.

Let MO $= b$, FB $= c$, SR $= d$, and FD $= q$, alſo GV $=$ VT $= x$; then (by Prop. 11.) $x : b$ :: $\overline{FT}^2 : \overline{FM}^2$ :: (by ſimilar Triangles) $y^2 : c^2$; and $x : d$ :: $\overline{FT}^2 : \overline{FS}^2$ :: (by ſimilar Triangles) $y^2 : q^2$; therefore, by Equality, $b : d :: c^2 : q^2$, or MO : SR :: $\overline{FB}^2 : \overline{FD}^2$. Q. E. D.

PROPOSITION XXII.

If from any Point in the Tangent, a right Line be drawn parallel to the Axe, meeting an Ordinate; the Rectangle of the Parameter of the Axe, into the external Part of that Line, will be equal to the Square of the Segment of the Ordinate, intercepted between that Line and the Point of Contact; that is, $p \times \text{MO} = \overline{FB}^2$, or $p \times \overline{RS} = \overline{FD}^2$.

DEMON-

DEMONSTRATION.

By the laſt Propoſition, $\frac{c^2 x}{b} = y^2 =$ (by Prop. 1.) px; alſo $\frac{q^2 x}{d} = y^2 = px$; therefore $pb = c^2$, and $dp = q^2$; that is, $p \times MO = \overline{FB}^2$, and $p \times RS = \overline{FD}^2$. Q. E. D.

PROPOSITION XXIII.

If FP touch the Parabola in F, and if from any Point S, in the Tangent, a right Line SD be drawn parallel to the Axe, cutting another right Line FC, drawn from the Point of Contact any how within the Curve; then the Curve will cut the firſt Line, in the ſame Proportion, as the firſt Line cuts the ſecond; that is, SR : RD :: FD : DC.

DEMONSTRATION.

Draw PC parallel to SD, and let $CP = r$, $RS = c$, $FS = d$, $RD = p$, $PS = m$, $FD = g$, and $DC = h$; then $c : r :: d^2 : \overline{d+m}^2 ::$ (by ſimilar Triangles) $g^2 : \overline{g+h}^2$, alſo (by ſimilar Triangles) $r : g+h :: c+p : g$; therefore $\frac{cg^2 + 2cgh + ch^2}{g^2} = r = \frac{cg + ch + pg + ph}{g}$, and conſequently $cgh + ch^2 = pg^2 + pgh$, or $ch \times \overline{g+h} = pg \times \overline{g+h}$

$\times\overline{g+h}$; therefore $ch=pg$, or $c:p::g:h$; that is, SR : RD :: FD : DC. Q. E. D.

PROPOSITION XXIV.

As the Abſciſſa, is to any Diameter terminated by the Ordinate, ſo is the Square of that Ordinate, to the Rectangle of the Parts of the Ordinate made by that Diameter; that is, VG : OB :: $\overline{FG}^2$: FB × BC.

DEMONSTRATION.

Let OB $=m$, FB $=c$, BC $=r$, MO $=b$; then (by Prop. 21) $x:b::y^2:c^2$, and (by the laſt Prop.) $b:m::c:r$; therefore $\frac{c^2x}{y^2}=b=\frac{mc}{r}$, and $rcx=my^2$, or $x:m::y^2:rc$; that is, VG : OB :: $\overline{FG}^2$: FB × BC. Q. E. D.

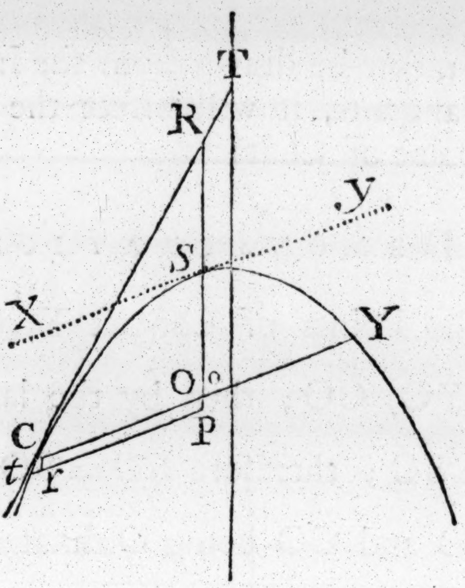
COROLLARY.

Hence it is manifeſt, OB : RD :: FB × BC : FD × DC; becauſe (by this Prop.) OB : VG :: FB × BC : $\overline{FG}^2$, and RD : VG :: FD × DC : $\overline{FG}^2$; conſequently OB : RD :: FB × BC : FD × DC.

PROPOSITION XXV.

If a Tangent cut any Diameter produced, and from the Point of Contact, an Ordinate be drawn to that Diameter; then the Diſtance (in the

Diameter produced) between the Vertex and Interſection of the Tangent, will be equal to the Abſciſſa of the Ordinate; that is, $RS=RO$.



DEMONSTRATION.

Let $OS=x$, $Cr=OP=n$, $tr=m$, $RS=a$, $OC=y$, and then P*t*, which is ſuppoſed to be indefinitely near to OC, will be $=m+y$, and $SP=n+x$; whence, by ſimilar Triangles, $m:n::y:a+x$; therefore $a+x=\frac{n\times y}{m}$, and (by Prop. 11.) $p\times SO=\overline{OC}^2$, alſo $p\times SP=\overline{Pt}^2$; that is, $px=y^2$ and $pn+px=m^2+2my+y^2$; therefore $px=y^2+2my-pn$; conſequently $y^2=y^2+2my-pn$, or $2my=pn$, and $n=\frac{2my}{p}$; let this Value be ſubſtituted for n in the firſt Equation, and it will become $x+a=\frac{2my}{p}\times\frac{y}{m}=\frac{2y^2}{p}=\frac{2px}{p}=2x$;

$= 2x$; conſequently $a = 2x - x = x$; or RS $=$ SO.

PROPOSITION XXVI.

If a Diameter be drawn from the Interſection of any two Tangents, it will biſect the Line, which joins the Point of Contact.

DEMONSTRATION.

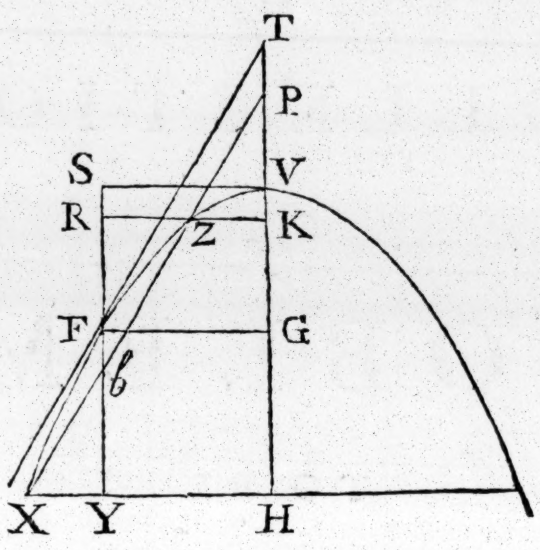
From the Points of Contact Y, C, draw the Ordinates YO, CO; then by the laſt RS $=$ SO and RS $=$ SO; therefore SO $=$ SO, and conſequently YO and CO being Ordinates to the ſame Diameter and Abſciſſa are equal, and in the ſame right Line. Q. E. D.

COROLLARY.

Hence another Method of drawing Tangents to a Parabola, from any Point without the Curve, may be deduced; for from the given Point R, draw the Diameter RP, and take therein, from the Vertex S, the Abſciſſa SO equal to the external Part RS; then through the Extremity of the Abſciſſa O, draw the right Line CY parallel to the Tangent XY, at the Vertex of that Diameter; and the Extremities of that Line will be Points in the Curve, in which Lines drawn from the given Point, will touch the Curve.

PROPOSITION XXVII.

If, from the Extremity of any Ordinate (Xb) to a Diameter, a right Line (as XY) be drawn at right Angles to the Diameter; then the Diſtance (XY) in that Line from the Extremity of the Ordinate to the Diameter, will be a Mean-proportional between the Parameter of the Axe, and the Abſciſſa of that Ordinate; that is, $p : \mathrm{XY} :: \mathrm{XY} : \mathrm{F}b$.



DEMONSTRATION.

Put $\mathrm{XY} = a$, $\mathrm{F}b = \mathrm{X}$; then (by Prop. 1.) $\overline{\mathrm{FG}}^2 = px$, and (by Prop. 5.) $\overline{\mathrm{GT}}^2 = 4x^2$; therefore (by 47. *Eu.* 1.) $\overline{\mathrm{FT}}^2 = px + 4x^2 = \overline{p + 4x} \times x$; but (by Prop. 12.) $\overline{\mathrm{X}b}^2 = \overline{p + 4x} \times \mathrm{X}$, and (from ſimilar Triangles) $\overline{\mathrm{FT}}^2 : \overline{\mathrm{FG}}^2 :: \overline{\mathrm{X}b}^2 : \overline{\mathrm{XY}}^2$; that is, $\overline{p + 4x} \times x : px :: \overline{p + 4x} \times \mathrm{X} : a^2$; therefore $a^2 = px$ and $p : a :: a : \mathrm{X}$, or $p : \mathrm{XY} :: \mathrm{XY} : \mathrm{F}b$. Q. E. D.

The End of the PARABOLA.

A
COLLECTION
OF
PROBLEMS,
TO BE
Anſwered in the next NUMBER.

PROBLEM I. *R.*

A Certain Number of Perſons agreed to give Sixpence a-piece to a Waterman, to carry them from *London* to *Graveſend*, on Condition, that if others were taken in by the Way, they ſhould pay the ſame Price, and that of the Money thence ariſing, one Half ſhould go to the Waterman, and the other Half to be equally divided among the firſt Perſons; now they happened to take in, one fourth Part of their Number, and three over, by Means whereof they only paid Fivepence each; What Number were there in Company at firſt?

PROBLEM II. *D.*

If a Cubic Foot of dry *Engliſh* Oak be put into a ſufficient Quantity of freſh Water, how much of it will be immerged, and how much emerge, and what Weight muſt be laid on it, to make it level with the Water's Surface?

PROBLEM III. *L.*

How high muſt an Eye be elevated above the Earth's Surface, to ſee two fifth Parts thereof; allowing the Earth to be ſpherical, and its Circumference 25020 Miles?

PROBLEM IV. *T.*

Having Occaſion to take the Declination of a Wall, and not being provided with proper Inſtruments for taking Altitudes, I had Recourſe to the following Method: I obſerved, by laying my Eye cloſe to the Edge of the Wall, the exact Time that the fixed Star *Pollux* came into the Plane of the Wall; and 56′ 10″ after (according to the Time ſhewn by a good Pendulum Clock) obſerved *Sirius* to paſs through the Plane. From theſe Obſervations the Plane's Declination may be determined, and is here required; the Latitude of the Place being 51° : 10′ North.

PROBLEM V. *E—— H——.*

In what Latitude North may an erect declining Dial be erected South Eaſterly; the Stile's Height; the Diſtance of the Subſtile from the Meridian, and the Plane's Declination whereof are equal to each other? 'Tis alſo required to know what Hours on *May* the 10th, 1745, the Sun comes on, and goes off the Plane.

PROBLEM VI. *I—— B——.*

At a certain Place in North Latitude, the Sun was obſerved to riſe exactly at 3 H. 58 Min. and at Six o'Clock his Altitude was taken the ſame Morning, and found to be 15° : 20′, his Declination being then North; required, the Latitude where, and Day of the Year when, theſe Obſervations were made.

PROBLEM VII. *R.*

Two Circles touching each other inwardly being given; to deſcribe a third Circle, that ſhall touch both the former, and alſo the right Line paſſing through their Centers.

PROBLEM VIII. *R.*

To draw a Chord, through a given Point, within a given Circle; the Parts whereof intercepted by that Point and the Periphery, may obtain a given Ratio.

PROBLEM IX. *L.*

The two Sides of any plane Triangle, and a right Line drawn from the Vertex, to the Middle of the Baſe being given; to determine the Triangle.

PROBLEM X. *N.*

The Perpendicular, the Line biſecting the vertical Angle, and the Line biſecting the Baſe of any plane Triangle being given; to determine the Triangle.

PROBLEM XI. *N.*

The Line biſecting the Baſe; the vertical Angle, and the Angle which the ſaid biſecting Line makes with the Baſe being given; to deſcribe the Triangle.

PROBLEM XII. *John Turner, London.*

To find a Point, in the Side of a given Square produced, from whence if a Line be drawn to the oppoſite Angle, the Part thereof intercepted between that Point, and the neareſt Side of the Square, ſhall be of a given Length; to be conſtructed and demonſtrated geometrically.

PROBLEM XIII. *R.*

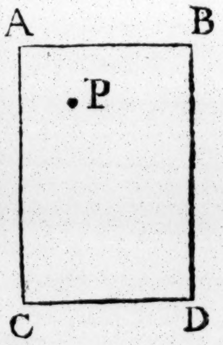
To determine a Point C, in a given right Line DF, from whence if two other right Lines AC and BC be drawn, to two given Points A and B, they ſhall comprehend an Angle ACB equal to a given Angle D.

PROBLEM XIV. *R.*

If from the Extremities of the Baſe of any plane Triangle, two right Lines be drawn, interſecting each other in the Perpendicular, and terminating in the oppoſite Sides; right Lines drawn from thence, to where the Perpendicular meets the Baſe, will make Angles with the Baſe equal to each other; *Quære*, the Demonſtration.

PROBLEM XV. *S—— A——.*

Let ABCD repreſent a given rectangular Billiard Table; it is required to find in what Direction, a Ball from the given Point P muſt be ſtruck; ſo that after three Reflections, it ſhall fall into the Purſe at the Angle B.



PROBLEM XVI. *R.*

The Difference of the Diameters of two Circles (ABE and BCD) touching each other inwardly (in B) being given; it is required to draw from (A) the other Extremity of the greater Circle, a Tangent to the leſſer, terminating in the Periphery of the greater; the Part whereof (DE) intercepted between the Point of Contact (D) and that Periphery may be of a given Length.

PROBLEM XVII. *T.*

To draw a right Line through a given Point, terminating in two right Lines given by Poſition, ſo as to be the ſhorteſt poſſible.

PROBLEM XVIII. *D.*

The Sum of the Diameters of two Circles touching each other outwardly being given, alſo the Length of the mixed Line circumſcribing them both; to find the Diameter of each.

PROBLEM XIX. *John Turner, London.*

If upon the principal Axe of any Curve, defined by the Equation $y = \dfrac{\overline{ax - x^2}|^{\frac{3}{4}} \times a^5 x^2}{\overline{a + x}|^9}$, a Semicircle be deſcribed; it is required to exhibit in finite Terms, the exact Ratio between that Semicircle, and the whole curvilineal Space included between the ſaid Curve and its Axis.

PROBLEM XX. *T.*

If the Earth be ſuppoſed to revolve round its Axis in 38 Minutes, and by Means thereof be projected two Bodies from off its Surface; the one in Latitude 52 Degrees, and the other under the Equator, with Velocities reſpectively as the Places from whence they are projected; it is required to determine the Trajectory of the former; and if it returns, its periodic Time, and where its Revolution will end; and likewiſe how far the latter will be from the Earth's Center in 6 Hours Time, allowing the Earth to be ſpherical, its Circumference 25020 *Engliſh* Miles.

N. B. This Problem was propoſed in the *Ladies Diary* for the Year 1736; the Solution whereof ſeveral attempted, in the ſucceeding Diary; but had not the deſired Succeſs.

The END.

THE

Mathematician.

DISSERTATION II.

Upon the Progreſs *and* Improvement *of* GEOMETRY.

THE former Diſſertation in Nº I. having been concluded a little too abruptly, becauſe at the Time of its Publication we thought to ſay no more of it; but its having met with a candid Reception, has made us think it expedient to reſume and purſue the Subject more minutely; and particularly with a Regard to the Improvements that Geometry has received from thoſe its illuſtrious Sons, *Wallis*, *Barrow*, *Newton*, &c.

Since the hiſtorical Part of the Invention of Fluxions has been treated of by ſeveral Hands, we ſhall in the following Pages, after mentioning the various Steps previous and preparatory thereto, endeavour to give as plain and clear an Account of that admirable Doctrine as the Nature of the Thing and our Abilities will admit, in order to evince, that it is truly and properly ſcientific; as will appear from the Accuracy and Reaſonableneſs of its Principles, and the Juſtneſs of its Rules, notwithſtanding it has been of late ſo much controverted.

To what has already been ſaid of Dr. *Wallis*'s Improvements may be added, that he found his Method of ſumming up Series's, to fail him in the Caſe of thoſe Progreſſions, whoſe Terms were the Roots of the Sums or Differences of ſimple Terms, called Roots univerſal, ſuch as $\sqrt{r^2 - 0a^2}$, $\sqrt{r^2 - 1a^2}$, $\sqrt{r^2 - 4a^2}$, $\sqrt{r^2 - 9a^2}$, $\sqrt{r^2 - 16a^2}$, &c. $\sqrt{r^2 - r^2}$, which he calls a Series of Terms in the ſubduplicate Ratio of a Series of Equals, leſſened by a Series of Secundans or Squares: Where, if *r* ſtand for the Radius of a Circle, and *a* for any of the indefinitely ſmall and equal Diſtances of the Ordinates in the Quadrant of the Circle, beginning at its Center, ſuch Quadrant is equal to the Sum of all the Terms of this Progreſſion; as he ſhews in Prop. 121 *Arith. Infinit.* They being the right Sines of which the Quadrant is compoſed; each of which are known to be a mean Proportional between $\overline{r + 0a}$ and $\overline{r - 0a}$, and between $\overline{r + 1a}$ and $\overline{r - 1a}$, and between $\overline{r + 2a}$ and $\overline{r - 2a}$, and ſo on. The ſame Series with the Sign of the ſecond Term under the Vinculum, changed into its Oppoſite, that is, $\sqrt{r^2 + 0a^2}$, $\sqrt{r^2 + 1a2}$, $\sqrt{r^2 + 4a^2}$, $\sqrt{r^2 + 9a^2}$, *&c.* to $r^2 + r^2$, being ſummed up, would give the Quadrature of the equilateral Hyperbola, ſuppoſing *r* to ſtand for the ſemitranſverſe Diameter, and *a* any of the indefinitely ſmall and equal Diſtances on the Aſſymptote, from the Center of the tranſverſe Diameter, then, as is evident from Conics, the Ordinates compoſing the hyperbolic Space will be expreſſed by the laſt Progreſſion. He likewiſe ſhews two other Series's in Prop. 165, of the ſame Book; by the ſumming up of whoſe Terms, the Quadrature of the Circle and Hyperbola would be found, but labouring under the ſame Inconveniency. Here the Doctor ſtuck, and owns *Hic Labor, hoc Opus eſt.* However, being very ſolicitous to do ſomething towards the Quradrature of the Circle, which was his principal View when he engaged in the Proſecution of theſe Enquiries, (as he tells us in the Preface of that Work) he thought upon another Method, which he calls the Interpolation of Series; by which he means, a Method of diſcovering certain intermediate Terms of a regular Series or Progreſſion, by conſidering the Properties of the Progreſſion, and the Relations of the Terms to each other. Of this he gives ſeveral Inſtances for finding the Area of the Circle; whereof this is one: In the Progreſſion $1, \frac{3}{2}, \frac{15}{8}, \frac{105}{48}$, *&c.* Whoſe Terms are produced by the continual Multiplication of $1 \times \frac{3}{2} \times \frac{5}{4} \times \frac{7}{6}$, *&c.* to find the intermediate Term betwixt 1 and $\frac{3}{2}$. But the Reſult of his Enquiry was, that the Value of ſuch intermediate Term cannot be adequately expreſſed by any received Way of Notation; which is nothing more ſtrange than that $\surd 2$, or any other ſurd Root, is not capable of being adequately expreſſed in that Way: But ſince the Value of $\surd 2$, or any other Surd, may be expreſſed approximately by the common Notation; ſo likewiſe he found, that the approximate Value of the Square, or the Circle's Area, was $1 \times \frac{3 \times 3 \times 5 \times 5 \times 7 \times 7,\ \&c.}{2 \times 4 \times 4 \times 6 \times 6 \times 8,\ \&c.}$ *in infinitum.* Notwithſtanding this Diſappointment, Dr. *Wallis* opened the Gate into a wide Field of geometrical Knowledge: For, in his arithmetical Works, publiſhed in 1657, he firſt reduced the Fraction $\frac{A}{1-R}$ by a perpetual Diviſion into an infinite Series $A + AR + AR^2 + AR^3 + AR^4$, &c.

Mr. *James Gregory*, in his *True Quadrature of the Circle and Hyperbola*, publiſhed in the Year 1667, gave a converging Series, or at leaſt ſhewed the Conſtitution of a converging Series, which proceeds by Pairs of Terms; ſuch, that the Difference between any Pair is greater than the Difference between the next ſubſequent Pair; and that after the ſame Manner that the ſecond Pair is formed by analytical Operation from the firſt, in like Manner is the third formed from the ſecond, and ſo on; by which the Difference continually leſſening, becomes leſs than any given Quantity; and the Series being ſuppoſed to be continued *in infinitum*, that Difference quite vaniſhes, and the two Terms become equal, either of which is the Quantity ſought; whereby you may approach to the Area of the Hyperbola, as well as that of the Circle, as near as you pleaſe.

In *April*, next Year, Lord Viſcount *Brounker* publiſhed an infinite Series for the ſame Purpoſe, which is that called the *Newtonian*, being now generally uſed; where the Aggregate of all the Terms, infinite in Number, is equal to the Quantity ſought; and the greater the Number of them taken from the Beginning, is, the nearer doth their Aggregate approach to the Quantity ſought. The ſame Year Mr. *Nicholas Mercator*, an *Holſatian* by Birth, but who had ſpent a great Part of his Life in *England*, publiſhed his *Logarithmotechnia*; in which he ſhewed how Lord Viſcount *Brounker*'s Series might be found, by reducing a complex Fraction to an infinite Series of ſimple Terms by Di-

Diviſion.; which was but a ſmall Improvement upon what Dr. *Wallis* had ſhewn in his *Math. Univ.* cap. 33. with reſpect to Diviſion. *Mercator* then had no more to do but to apply the ſame Perſon's Method of Quradratures in his *Arith. Inf.* for ſquaring thoſe ſeveral ſimple Terms. Upon the Publication of the *Logarithmotechnia* Dr. *Wallis* illuſtrated the Diſcovery; and gave another infinite Series for the ſame Purpoſe, in the *Philoſophical Tranſactions* for *Auguſt* 1668. Towards the End of the ſame Year, Mr. *James Gregory* publiſhed his *Exercitationes Geometricæ*; in which he promoted and enlarged *Mercator*'s Diſcovery, and gave a geometrical Demonſtration of it, by Means of ſumming up the Secants of a circular Arch.

We muſt not omit to obſerve a very remarkable Concluſion that Dr. *Wallis* obtained in his *Arith. Inf.* by his gradual Method of Induction. He conſidered thoſe Progreſſions as conſiſting of an infinite Number of Terms; and having adopted *Cavalerius*'s Method of Indiviſibles, the Elementa of which geometrical Figures were by that Method compoſed, were naturally repreſented by the Terms of thoſe Progreſſions, *viz.* the laſt Term, which repreſented the loweſt Ordinate of a Curve, being ſtill finite; and the intermediate Terms from 0 to the laſt, being infinite in Number, repreſented Ordinates applied to the Axis at infinitely ſmall and equal Diſtances, betwixt the Vertex and loweſt Ordinate; or, perhaps, theſe Terms repreſented any other Lines, right or curved; or any plain or curved Surfaces, in the Caſe of Solids, which were proportional to them: Now nothing was wanted but a Method of ſumming them up. At laſt, the Doctor found this moſt general and comprehenſive Propoſition, which contains the Subſtance of his whole Book, *viz.* The Sum of all the Terms of any

any Series of the *(m)* Powers of Quantities in arithmetical Progreſſion from 0, is equal to the Product of the laſt Term, by the Number of Terms; and this divided by the Index *(m)* plus 1. Which amounts to this; ſuppoſing 0, 1, 2, 3, 4, &c. x, to be an arithmetical Progreſſion, conſiſting of an infinite Number of Terms, in the natural Order of Numbers, having the laſt Term x; and let 0^m, 1^m, 2^m, 3^m, 4^m, &c. x^m, be a Progreſſion of Terms, which are any the ſame Power, Root, or Dimenſion, whatſoever of the former Terms, whoſe Exponent is denoted by *(m)*; then ſhall the Sum of this laſt Series or Progreſſion of Powers, Roots, *&c.* be equal to $\frac{x^{m+1}}{m+1}$. Here Dr. *Wallis* may be ſaid to end, and Sir *Iſaac Newton* to begin his Improvements; for this is in Subſtance the ſame with his firſt Rule in his Analyſis, and upon which he builds his Quadratures of Curves. *Wallis* demonſtrated this general Propoſition by an Induction from ſeveral particular ones, collected at laſt into one by a Table of Caſes: But Sir *Iſaac* reduced all the Caſes into one, and demonſtrated it univerſally by Means of an indefinite Index, which he firſt introduced into analytical Operations.

When theſe Diſcoveries of Dr. *Wallis* (which were indeed very noble and uſeful, and, in Point of Generality and Extent, far exceeding every Thing that had been before done by others in the Geometry of Curve Lines) were made public, it was objected to him by Mr. *Fermat* and others, that however valuable his Diſcoveries were, and true in themſelves, yet the Demonſtrations of them in the Way of Induction, did not come up to that Accuracy which a geometrical Subject required, and which the ancient Geometricians had all along obſerved in their Performances. To which Dr.

Wallis made Anſwer, (even as he had remarked in the *Arithmetic of Infinites* itſelf) That he did not ſo much deſign to demonſtrate his Diſcoveries, as to lay open to others the Method he uſed in making them, which the Antients purpoſely concealed; that, notwithſtanding, he thought the Proof, by Way of Induction, was ſatisfying and convincing; that it would be an eaſy Matter for any Perſon, moderately ſkilled in Geometry, to demonſtrate theſe Things, with all the Pomp and Apparatus made uſe of by the Antients; but that was a Labour he never deſigned to undertake. However, to give ſome Satisfaction in this Matter, he ſhews, by ſome Examples, in the 78th Chapter of his Algebra, how the Propoſitions he had diſcovered might be demonſtrated after the Manner of the Antients, in Imitation of what had been done by *Archimedes* in the 10th and 11th Propoſitions of his Treatiſe of ſpiral Lines; wherein *Archimedes* demonſtrates what is the Sum of all the Terms of a Progreſſion of Squares, whoſe Sides conſtitute an arithmetical Progreſſion of Lines, having the common Difference equal to the firſt Term, when compared with ſo many Times the greateſt Square; and the Limits betwixt which the Sum of the Terms of ſuch a Progreſſion is contained, although the common Difference of their Sides be not the ſame as the firſt of them; which he applies to the finding the Relation of the ſpiral Spaces to the circular Sectors; even as Dr. *Wallis*, by proſecuting this Affair to a much greater Length, ſhews how to find not only the Sum of all the Squares, but the Sum of any Powers or Roots whatſoever of an arithmetical Progreſſion; and thereby to compare infinite Numbers of curvilinear Areas with right-lined Figures, and with one another. And truly, when one attentively conſiders this elaborate Treatiſe of *Archimedes*, and the other Works of that ſubtle and penetrating Genius, one cannot help ſeeing Dr. *Wallis*'s *Arithmetic of Infinites*, Mr. *Gregory*'s *Method of Inſcription and Circumſcription of Polygons*, and even Sir *Iſaac*'s *Method of prime and ultimate Ratio's*, beginning to diſcover themſelves, as it were, in Embryo, in order to be brought forth afterwards to open Light and Perfection.

If it ſhould be here objected, that ſince all the Terms of an infinite Series are unaſſignable, it is impoſſible to determine their Sum; to obviate this, let it be conſidered, that a Number actually infinite, (*i. e.* all whoſe Units can be actually aſſigned, and yet is without Limits) is a plain Contradiction to all our Ideas about Numbers; for whatever Number we can actually conceive, or have any proper Idea of, is and muſt be always determinate and finite; ſo that a greater after it may be aſſigned, and a greater after this, and ſo on, without a Poſſibility of ever coming to an End of the Addition or Increaſe of Numbers aſſignable; which Inexhauſtibility or endleſs Progreſſion in the Nature of Numbers, is all that we can diſtinctly underſtand by the Infinity of Numbers; and therefore to ſay, that the Number of any Things is infinite, is not ſaying, that we comprehend their Number, but indeed the contrary; the only Thing poſitive in this Propoſition being this, *viz.* that the Number of theſe Things is greater than any Number which we can actually conceive and aſſign. We eaſily conceive, that a finite Magnitude may become greater and greater without End, or that no Termination or Limit can be aſſigned of the Increaſe which it may admit; but we do not therefore clearly conceive Magnitude increaſed an infinite Number of Times. Mr. *Locke*, who wrote his excellent Eſſay, that we might diſcover how far the Powers of the Underſtanding reach, to what Things they are in any De-

Degree proportionate, and where they fail us, he acknowledges, that we eaſily form an Idea of the Infinity of Number; to the End of whoſe Addition there is no Approach: But he diſtinguiſhes between this and the Idea of an infinite Number; and ſubjoins, that how clear ſoever our Idea of the Infinity of Number may be, there is nothing more evident than the Abſurdity of the actual Idea of an infinite Number. He likewiſe obſerves, that whilſt Men talk and diſpute of infinite Magnitudes, as if they had as compleat and poſitive Ideas of them as they have of the Names they uſe for them, or as they have of a Yard, an Hour, or any other determinate Quantity, it is no Wonder, if the incomprehenſible Nature of the Thing they diſcourſe of, or reaſon about, leads them into Perplexities and Contradictions, and their Minds be overlaid by an Object too large and mighty to be ſurveyed and managed by them. Mathematicians, indeed, abridge their Computations by the Suppoſition of Infinites; but when they pretend to treat them on a Level with finite Quantities, they are ſometimes led into ſuch Doctrines as verify the Obſervation of this judicious Author.

We cannot apply to an infinite Series the common Notion of a Sum, *viz.* a Collection of ſeveral particular Numbers, that are joined and added together, one after another; for this ſuppoſes, that theſe Particulars are all known and determined; whereas, the Terms of an infinite Series cannot be all ſeparately aſſigned, there being no End in the Numeration of its Parts, and therefore it can have no Sum in this Senſe. But again, conſider, that the Idea of an infinite Series conſiſts of two Parts, *viz.* the Idea of ſomething poſitive and determined, in ſo far as we conceive the Series to be actually carried on; and the Idea of an inexhauſtible Remainder ſtill behind, or an endleſs

Addition of Terms that can be made to it, one after another; hence we may conceive it as a Whole of its own Kind, and therefore may be ſaid to have a real Value, whether that be determinable or not. Now, in ſome infinite Series, this Value is finite or limited, *i. e.* a Number is aſſignable, beyond which the Sum of no aſſignable Number of Terms of the Series can ever reach, nor indeed ever be equal to it; yet may approach to it in ſuch a Manner, as to want leſs than any aſſignable Difference; and this we may call the Value or Sum of the Series; not as being a Number found by the common Method of Addition, but as being ſuch a Limitation of the Value of the Series, taken in all its infinite Capacity, that if it were poſſible to add them all one after another, the Sum would be equal to that Number. Again, in other Series, the Value has no Limitation; and we may expreſs this by ſaying, The Sum of the Series is infinitely great; which indeed ſignifies no more than that it has no determinate and aſſignable Value; and that the Series may be carried ſuch a Length, as its Sum, ſo far, ſhall be greater than any given Number. In ſhort, in the firſt Caſe, we affirm, there is a Sum; yet not a Sum taken in the common Senſe; in the other Caſe, we plainly deny a determinate Sum in any Senſe. According to the common Rule for ſumming up a finite Progreſſion of a geometric Series decreaſing, where r is the Ratio, l the firſt Term, and A the leaſt, the Sum is $= \frac{rl - A.}{r - 1}$ If we ſuppoſe A the leſſer Extream actually decreaſed to 0, then the Sum of the whole infinite Series is $= \frac{rl}{r - 1}$: For it is demonſtrable, that the Sum of no aſſignable Number of Terms of the Series can ever be equal to that Quotient; and yet no

Number leſs than it, is equal to the Value of the Series. And whatever Conſequences follow from the Suppoſition of $\frac{rl}{r-1}$, being the true and adequate Value of the Series, taken in all its infinite Capacity, as if the Whole were actually determined and added together, can never be the Occaſion of any aſſignable Error, in any Operation or Demonſtration, where it is uſed in that Senſe; becauſe, if you ſay it exceeds that adequate Value, yet it is demonſtrable, that this Exceſs muſt be leſs than any aſſignable Difference, which is in Effect no Difference; and ſo the conſequent Error, will be in Effect no Error: For if any Error can happen from $\frac{rl}{r-1}$ being greater than it ought to be, to repreſent the compleat Value of the infinite Series, that Error depends upon the Exceſs of $\frac{rl}{r-1}$ over that compleat Value; but this Exceſs being unaſſignable, that conſequent Error muſt be ſo too; becauſe ſtill the leſs the Exceſs is, the leſs will the Error be that depends upon it. For which Reaſon, we may juſtly enough look upon $\frac{rl}{r-1}$ as expreſſing the adequate Value of the infinite Series: But we are farther ſatisfied of the Reaſonableneſs of this, by finding, in Fact, that a finite Quantity does actually convert into an infinite Series, as appears in the Caſe of infinite Decimals. *E. g.* $\frac{2}{3} = .6666$, *&c.* is plainly a geometric Series in the Decimal Scale, from $\frac{6}{10}$ in the continual Ratio of 10 to 1; and may very juſtly be expreſſed by an infinite Series of the Reciprocals of the Powers of 10, which is the Root of the Scale; and if 10 be denoted by x, will be $6x^{-1} + 6x^{-2} + 6x^{-3} + 6x^{-4}$, *&c.* And, reverſely, if we take

this

this Series, and find its Sum by the preceding Expreſſion, it comes to the ſame $\frac{2}{3}$; for $l = 6x^{-1} = \frac{6}{10}$, $r = x = 10$, therefore $rl = \frac{60}{10} = 6$; and $r - 1 = 9$, whence $\frac{rl}{r - 1} = \frac{6}{9} = \frac{2}{3}$. By the ſame Artifice, uniformly carried on, may all Decimals, ſimple or mixed, be expreſſed, provided we aſſume the Co-efficients 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, as Occaſion ſhall require, to its proper Term in the Series; thus the mixt Number 526.384, becomes $5x^{3} + 2x^{2} + 6x^{0} + 3x^{-1} + 8x^{-2} + 4x^{-3}$.

The like may be done by any other Scale as well as the Decimal Scale, or by admitting any other Root beſides 10, to be the Root of our Arithmetic; for the Root 10 was an arbitrary Number, and, at firſt, aſſumed by Chance, without any previous Conſideration of the Nature of the Thing. Other Numbers, perhaps, may be aſſigned, which would have been more convenient, and which have a better Claim for being the Root of the vulgar Scale of Arithmetic. The Sexagenary or Sexageſimal Scale obtains among Aſtronomers; and expreſſes all poſſible Numbers, Integers, or Fractions, rational or ſurd, by the Powers of 60; and certain numeral Co-efficients, not exceeding 59. Any Number whatever, whether Integer or Fraction, may be made the Root of a particular Scale of Arithmetic, and all conceiveable Numbers may be expreſſed or computed by that Scale, at leaſt, by Approximation, admitting only of integral and affirmative Co-efficients, whoſe Number (including the Cypher 0) need not be greater than the Root. And it appears by the Inſtance above, that ſome Numbers may be expreſſed by a finite Number of Terms in one Scale, which in another cannot be expreſſed but by Approximation, or by a Progreſſion of Terms *in infinitum*. It further appears, that a Number computed by any one Scale, is eaſily reduced to any other Scale aſſigned, by ſubſtituting inſtead of the Root in one Scale, what is equivalent to it expreſſed by the Root of the other Scale. Thus to reduce Sexagenary Numbers to Decimals; becauſe 60, or X is $= 6 \times 10$; or $X = 6x$ therefore $X^2 = 36x^2$, $X^3 = 216x^3$, *&c.* by the Subſtitution of theſe we ſhall eaſily find the equivalent Decimal Number. All vulgar Fractions, and mixed Numbers, are, in ſome Meaſure, the Expreſſions of Numbers by a particular Scale; or making the Denominator of the Fraction to be the Root of a new Scale; thus $\frac{2}{3}$ is in Effect $0 \times 3^0 + 2 \times 3^{-1}$, and $8\frac{3}{5}$ is the ſame as $8 \times 5^0 + 3 \times 5^{-1}$, and ſo of others.

The Co-efficients in theſe Scales are not neceſſarily confined to be affirmative integer Numbers leſs than the Root, (though they ſhould be ſuch, if we would have the Scale to be regular;) but as Occaſion requires, they may be any Numbers whatſoever, affirmative or negative, Integers or Fractions. And, indeed, they generally come out promiſcuouſly in the Solution of Problems. Nor is it neceſſary, that the Indices of the Powers ſhould be always integral Numbers; but may be any regular arithmetical Progreſſion whatever; and the Powers themſelves either rational or irrational. For, ſuppoſing the Root of the Scale to be an indefinite or general Number, repreſented by x or y, &c. and aſſuming the general Co-efficients a, b, c, d, &c. which are Integers or Fractions, affirmative or negative, as it may happen; we may form ſuch a Series as this, $ax^4 + bx^3 + cx^2 + dx^1 + ex^0$; which will repreſent ſome certain Number expreſſed by the Scale, whoſe Root is x. If ſuch a Number proceeds *in infinitum*, then it is truly and properly called an infinite or converging Series; x being then ſuppoſed greater than Unity. Such, for Example, is $x + \frac{1}{2}x^{-1} + \frac{1}{3}x^{-2} + \frac{1}{4}x^{-3}$, *&c.*

&c. And it may have any deſcending arithmetical Progreſſion for its Indices, as $x^{m} + \frac{2}{3}x^{m-1} + \frac{3}{4}x^{m-2} + \frac{4}{5}x^{m-3}$, *&c.*

And thus we have been led by proper Gradations, that is, by arguing from what is well known and commonly received, *viz.* the Doctrine of decimal Fractions; to what before appeared to be difficult and obſcure, *viz.* the Knowledge of an univerſal or infinite Series. The great Similitude between the Nature and Operations of them, occaſions the former to be a convenient Illuſtration of the latter; for the chief Difference between theſe infinite Series's in decimal Arithmetic, and thoſe in the literal or ſpecious, is, that in the former there is only one Scale or Progreſſion of Terms, which varies in a decuple Ratio; and the Co-efficients are all poſitive Integers below 10; whereas, in the latter, which is of a more general and indefinite Nature, the Scales or Progreſſions may be infinitely varied, in a decuple, or any other Ratio whatever; and the numeral Co-efficients may be any Numbers integral or fractional, poſitive or negative, as hath been before obſerved. It is likewiſe evident, that the Operations in decimal Arithmetic, by which the Quotient or Quantity ſought is diſcovered, correſpond to the Operations relating to infinite Series in ſpecious Arithmetic; in this Reſpect, that every new Step of the Operation, in both Caſes, by which the conſtituent Parts of the Quotient are found, makes as great an Advance towards the Supply of what the Quotient is yet deficient, as the Nature of the Progreſſion will admit. But all general Series, which are commonly the Reſult in the higher Problems, muſt paſs by Subſtitution to particular Scales or Series; and theſe, when applied to Practice, muſt have their indefinite and determinate Characters, in order to be finally reduced to the Decimal Scale. And the

Art of finding ſuch general Series's, and then their Reduction to particular Scales, and laſt of all to the common Decimal Scale, is almoſt the Whole abſtruſer Parts of Analytics.

Our having dwelt ſo long upon the Nature of Series's here, will have its Uſe, by contracting what is hereafter to be ſaid, when we come to treat of Sir *Iſaac Newton*'s analytical Improvements, which were the Keys of his Operations in the ſublime Geometry.

So much for Dr. *Wallis*. The next Promoter of Geometry, with reſpect to Time, among our Countrymen, was Dr. *Barrow*, a Man of a penetrating Genius, and very indefatigable: He had amaſſed a large Magazine of Learning; and his general Character was, that whatever Subject he treated on, he exhauſted: He was a perfect Maſter of the ancient Geometry; and has obliged us with compendious, yet clear Demonſtrations of what is left of the geometrical Writings of *Euclid*, *Archimedes*, *Apollonius*, and *Theodoſius*. But the Advances he made in curve-lined Geometry, his own particular Improvements, are contained in his Lectures. He begins with treating on the *Generation of Magnitude*, which comprehends the Original of mathematical Hypotheſes. Magnitude may be produced various Ways, or conceived ſo to be; but the primary and chief among them is that performed by *Local Motion*, which all of them muſt in ſome Sort ſuppoſe; becauſe, without Motion, nothing can be generated or produced: So true is *Ariſtotle*'s Axiom, *viz.* He that is ignorant of Motion, is neceſſarily ignorant of Nature. What Mathematicians chiefly conſider in Motion are theſe two Properties, *viz.* The Mode of Lation, or Manner of Bearing; and the Quantity of the motive Force. From theſe Springs the Differences of Motions flow; but becauſe the Quantity of motive Force cannot be known without Time, the Doctor gives a long metaphyſical Account of the Nature of Time; which he defines to be, abſtractedly, *The Capacity or Poſſibility of the Continuance of any Thing in its own Being.* Towards the latter End whereof, he agrees with *Ariſtotle*, that we not only meaſure Motion by Time, but alſo Time by Motion; becauſe they determine each other: For in like Manner as we firſt of all meaſure a Space by ſome Magnitude, and declare it is ſo much; and afterwards, by Means of this Space, compute other Magnitudes correſpondent with it: So we firſt aſſume Time from ſome Motion, and afterwards judge thence of other Motions; which, in Reality, is no more than comparing ſome Motions with others, by the Aſſiſtance of Time; juſt as we inveſtigate the Ratio's of Magnitude by the Help of ſome Space. *E. g.* He who computes the Proportion of Motion by the Proportion of Time, does no more than get the ſaid Ratio of Motions from Clocks, Dials, or from the Proportion of ſolar Motions in the ſame Time. Again, becauſe Time is a Quantity uniformly extended, all whoſe Parts correſpond, either proportionally to the reſpective Parts of an equal Motion, or to the Parts of Spaces moved through with an unequal Motion; it may therefore be very aptly repreſented to our Minds, by any Magnitude alike in all its Parts; and eſpecially the moſt ſimple ones, ſuch as a ſtrait or circular Line; between which and Time there happens to be much Likeneſs and Analogy: For as Time conſiſts of Parts altogether ſimilar, it is reaſonable to conſider it as a Quantity endowed with one Dimenſion only; whether we imagine it to be made up, as it were, either of the ſimple Addition of riſing Moments, or of the continual Flux of one Moment; and for that Reaſon aſcribe only Length to it, and determine its Quantity by the Length of a Line paſſed over. As a Line is looked on to be the Trace of a Point moving forward, being in ſome Sort diviſible by a Point, and may be divided by Motion one Way, *viz.* as to Length; ſo Time may be conceived as the Trace of a Moment continually flowing; having ſome Kind of Diviſibility from an Inſtant, and from a ſucceſſive Flux, inaſmuch, as it can be divided ſome Way or other. And like as the Quantity of a Line conſiſts of but one Length following the Motion, ſo the Quantity of Time purſues but one Succeſſion ſtretched out, as it were, in Lengrh; which the Length of the Space moved over, ſhews and determines. Time may therefore always be expreſſed by a right Line; firſt, indeed, taken or laid down at Pleaſure; but whoſe Parts will exactly anſwer to the proportionable Parts of Time, as its Points do the reſpective Inſtants of Time, and will aptly ſerve to repreſent them.

The Doctor next proceeds to the effective Force of Time, being the ſame as what he before called the Motive Force by which Magnitudes are generated. He conſiders this as a Kind of Quantity, capable of Computation, like other Quantities: For it is plain from Experience, that when two moveable Bodies depart from the ſame Place along the ſame Line, the one moves a greater Space than the other in the ſame Time; the Reaſon of which can only be this, that that Body which moves ſwifteſt, is acted upon by a greater Force or motive Power; this Force therefore admitting of greater and leſſer Modifications, may be juſtly conceived as diviſible into any infinite or indefinite Parts; the leaſt of which is called Reſt, or the loweſt Degree of Velocity: Conſidering therefore the Thing abſolutely, in order to repreſent the Quantity of this Force juſtly to the Mind, we need only lay down ſome regular Magnitude in its Stead.

As a right Line is the moſt ſimple and perſpicuous of any, it is therefore the fitteſt to repreſent any Degree thereof. When this Force comes under a mathematical Conſideration, it is called Velocity; which is defined to be *That Power by which a moveable Body can paſs over a given Space in a given Time*; whence it follows, that every particular Quantity of any Velocity cannot be known, neither by the Space moved through only, nor by the Time ſingly, but may be found by Calculation from the Quantity of Space and Time together; as on the contrary, the Quantity of Time may be obtained from the Quantity of the Space and Velocity together: Nor does the Quantity of Space (ſo far as it can be known this Way by Motion) depend wholly upon the Quantity of a definite Velocity, or upon any aſſigned Time, but upon the conjoint Ratio of both. The Quantity of Space is found after the ſame Manner as we do that of a Superficies, by its Dimenſions; but the Quantities of Velocity and Time are found exactly after the ſame Manner as when a Superficies and one of its Dimenſions are given, we thereby find the other: For to every Moment of Time there anſwers ſome Degree of Velocity, which a moveable Body is then conceived to have; to which Degree ſome Length of the Space moved over anſwers. When Time flows equally, it will be moſt aptly repreſented by a right Line; and the ſeveral Degrees of Velocity, whether equal or unequal in each Inſtant, may be alſo expreſſed by right Lines; and becauſe theſe Degrees of Velocity do in every Moment of Time paſs over one another independently, and without Mixture; therefore, if right Lines parallel to each other, and horizontal, be drawn through all the Points of the perpendicular Line repreſenting the Time, the plain Superficies thence reſulting, will exactly repreſent the Aggregate of the De-

Degrees of Velocity; which Superficies having its Parts proportionable to the reſpective Parts of the Space moved through, may very well repreſent that Space. If the Velocities anſwering to each Inſtant of Time are equal, this Superficies will be a Parallelogram; if unequal, a Triangle: From the Properties of the former Figure are deduced all the Theorems of equable and uniform Motion; and from the latter, all thoſe which concern equally accelerated Motion. Moreover, if the Degrees of Velocity, in a continual Succeſſion from Reſt, throughout every Inſtant of Time, to a given Degree, be conceived to increaſe to it, or decreaſe from thence to Reſt, in the Progreſſion of the ſquare Numbers, the aggregatical Velocity, as well as the Space moved through, may moſt conveniently be repreſented by the Semiparabola; whoſe Vertex denotes Reſt, the ſeveral equal Parts of the Abſciſs, the given equal Times, and the Ordinates, the reſpective Degrees of Velocity, from a well known Property of the Parabola. In like Manner, any ſuppoſed Degrees of Velocity, any how increaſing or decreaſing continually, or interruptedly after any imaginable Way, may be truly and conveniently expreſſed by right Lines applied to that repreſenting the Time, keeping whatever Proportion any one is pleaſed to aſſign; ſo that knowing from thence the Meaſure of the repreſentative Space, the Quantity of Space moved through will be eaſily had, and the contrary; for it is eaſy to deduce Theorems, if any one knows rightly and congruouſly how to reduce Quantities of any Kind ſoever, ſubject to his Contemplation, to analogous Magnitudes.

Perhaps, this dry Account of theſe metaphyſical Subjects may ſeem tedious; but if it be conſidered, that hereupon are founded the Theories of the Deſcent of heavy Bodies, of Pendulums, and of Projectiles; the reducing of which to geometrical

Demonſtrations raiſed the famous *Galileo* to ſo high a Reputation, that he was ſaid to have added two new Sciences to the Mathematics; and when we further conſider, that the Doctrine of Fluxions is compriſed in two mechanical Problems, and that Mechanics, or the Doctrine of Motion, depends upon Computation of the Quantities of Time, Velocities, and Forces, it will then appear, that theſe Conſiderations directly lead towards Fluxions, and that it cannot be Time ill ſpent to conſider their Nature abſtractedly; unleſs we could be content to know the Manner of operating by them only, without contemplating the Reaſon of them in Theory, and knowing whether they are really ſcientific or no.

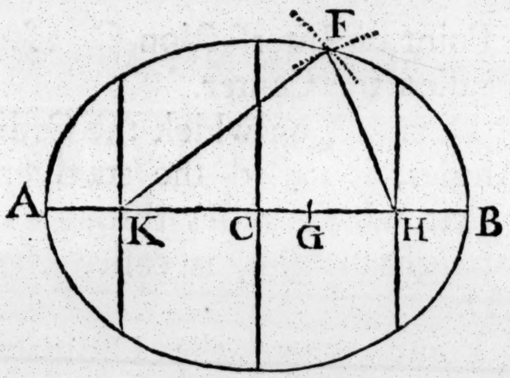
To be continued.

CONIC SECTIONS.

Of the ELLIPSE.

The GENESIS.

IF in any right Line, drawn upon a Plane, as AB, there be taken two Points K and H, equally remote from the Middle thereof, and in the ſame alſo be taken any other Point G, and from the former as Centers, with the Radii AG, BG reſpectively, be deſcribed two Areas; they will interſect each other in the Periphery of an Ellipſe, paſſing through the Extremities of the Line AB.



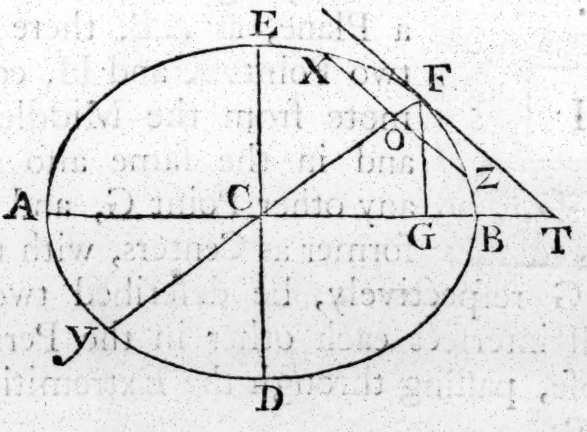
For, by Conſtruction, $KF = AG$, and $HF = GB$; therefore $HF + KF = AG + GB = AB$.

In like Manner an indefinite Number of Points may be found; through which, if a Curve Line be ſuppoſed drawn, it will comprehend a Space called an Ellipſe.

DEFINITIONS.

1. The Points H and K are called the Foci.

2. A Diameter is a right Line which paſſes through C, the Middle of AB, and biſects all Lines within the Curve, that are parallel to the Tangent touching its Vertex, and the Lines ſo biſected are called Ordinates to that Diameter; ſo FY is a Diameter; XO = OZ are Ordinates, being parallel to the Tangent touching the Curve in F, the Vertex of the Diameter.



3. The Point of Interſection C, of all the Diameters, is called the Center.

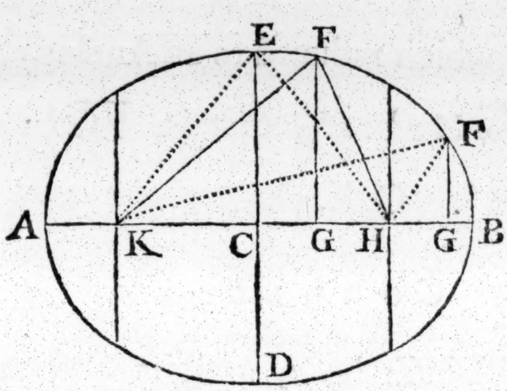
4. That Diameter on which the Ordinates ſtand at right Angles, is called the tranſverſe Axe, as AB; and that which paſſes through the Center, cutting it at right Angles, is called the conjugate Axe, as ED.

5. The Point, where the Ordinates interſect the Diameter, is called the Point of Application, as G and O.

6. The Segment of the Diameter, intercepted between the Vertex and the Point of Application, is called the Abſciſſa; as FO, OY; or BG, AG.

PROPOSITION I.

As the Square of any Ordinate to the tranſverſe Axe is to the Rectangle of the Abſciſſas which it divides, ſo is the Square of the Conjugate to the Square of the tranſverſe Axe.



DEMONSTRATION.

Let AC $= t$, CE $= c$, KC $= b$, CG $= x$, FG $= y$, and z equal to the Difference between the Line KF and the Semi-tranſverſe Axe AC; then KH $= 2b$, KG $= b + x$, and GH $= x - b$ or $b - x$, according as the Point G falls on this or that Side the Focus H; alſo, by the Geneſis, KF $= t + z$ and FH $= t - z$; whence (by *Eu.* 47. 1.) $\overline{HF}^2 = \overline{HG}^2 + \overline{GF}^2$; or $t^2 - 2tz + z^2 = b^2 - 2bx + x^2 + y^2$, and $KF^2 = \overline{KG}^2 + GF^2$;

or $t^2+2tz+z^2=b^2+2bx+x^2+y^2$; hence, the former of theſe Equations taken from the latter gives, $4tz=4bx$; therefore $z=\frac{4bx}{4t}=\frac{bx}{t}$; which being ſubſtituted in Place of z, in either of the foregoing Equations, there will come out $t^4+b^2x^2=t^2b^2+t^2x^2+t^2y^2$; but $b^2=t^2-c^2$, by the Geneſis; therefore $\overline{t^2-x^2}\times c^2=t^2y^2$; which reduced to an Analogy, gives $y^2:\overline{t+x}\times\overline{t-x}::c^2:t^2$; that is, $\overline{FG}^2:AG\times GB::\overline{DE}^2:\overline{AB}^2$. Q.E.D.

COROLLARY.

Let any Abſciſſa be x, and its Ordinate y, the tranſverſe Axis t, and the Conjugate c; (which Symbols repreſent the ſame Things in all the following Demonſtrations) then by this Theorem, $t^2:c^2::\overline{t-x}\times x:y^2$; or $t^2y^2=c^2tx-c^2x^2$; which generally is called the Equation of the Curve.

DEFINITION.

A third Proportional to the tranſverſe and conjugate Axis, is called the Parameter of the Axe; that is, if for the Parameter be put p, then $t:c::c:p$; therefore $tp=c^2$.

PROPOSITION II.

As the tranſverſe Axe, is to its Parameter, ſo is the Rectangle of any two Abſciſſas, to the Square of the Ordinate which divides them.

De-

DEMONSTRATION.

By the Definition of the Parameter $tp = c^2$, and by putting tp in the Equation of the Curve for c^2, a new Equation of the Curve will be produced in Terms of the Parameter, *&c.* viz. $ty^2 = tpx - px^2$; or $y^2 = \frac{p}{t} \times \overline{tx - x^2}$; therefore $t : p :: \overline{t - x} \times x : y^2$. Q. E. D.

COROLLARY.

As the Rectangle of any two Abſciſſas, is to the Square of the Ordinate which divides them, ſo is the Rectangle of any other two Abſciſſas, to the Square of the Ordinate which divides them. For (by this Prop.) $t - x \times x : y^2 :: t : p :: \overline{t - X} \times X : Y^2$.

PROPOSITION III.

The tranſverſe Axe into one fourth of its Parameter, is equal to the Rectangle of the greateſt and leaſt Diſtance of either Focus from the Vertex; that is, $\frac{1}{4}p \times AB = AH \times HB = BK \times KA$.

DEMONSTRATION.

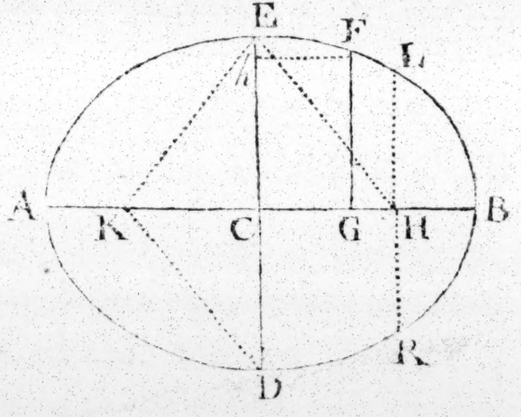
Let $HB = q$, then $HA = t - q$, and $CH = \frac{1}{2}t - q$: But $\overline{HE}^2 = \overline{EC}^2 + \overline{CH}^2$; that is, $\frac{1}{4}t^2 = \frac{1}{4}t^2 - tq + q^2 + \frac{1}{4}c^2$; or $\overline{t - q} \times q = \frac{1}{4}c^2 = \frac{1}{4}pt$; or $\frac{1}{4}p \times AB = AH \times HB$. Q. E. D.

COROLLARY.

The ſemi-conjugate Axe, is a mean Proportional between the greateſt and leaſt Diſtance of either Focus from the Vertexes: For ſince $\overline{t-q}\times q = \frac{1}{4}c^2$; therefore $t-q : \frac{1}{2}c :: \frac{1}{2}c : q$; that is, AH : CD :: CD : HB.

PROPOSITION IV.

The Parameter of the Axe, is double the Ordinate applied to the Focus.



DEMONSTRATION.

Let the focal Diſtance be q, and the Ordinate paſſing through the Focus y; then (by Prop. 2.) $t : p :: \overline{t-q}\times q : y^2$; but (by Prop. 3.) $\overline{t-q}\times q = \frac{1}{4}pt$; therefore $t : p :: \frac{1}{4}pt : \frac{1}{4}p^2 = y^2$, and $\frac{1}{2}p = y$; or $p = 2y$. Q. E. D.

PROPOSITION V.

The Diſtance between the Foci, is a mean Proportional between the Sum and Difference of the Tranſverſe and conjugate Axe; that is, AB + DE : KH :: KH : AB — DE.

DEMONSTRATION.

For KH put b; then $\overline{KD}^2 - \overline{CD}^2 = \overline{KC}^2$; that is, $\frac{1}{4}t^2 - \frac{1}{4}c^2 = \frac{1}{4}b^2$; or $t^2 - c^2 = b^2$; therefore $t + c : b :: b : t - c$; or AB + DE : KH :: KH : AB — DE. Q. E. D.

PROPOSITION VI.

A fourth Proportional to the Conjugate, Tranſverſe, and any Ordinate, is equal to a mean Proportional between the Abſciſſas of that Ordinate.

DEMONSTRATION.

Let the fourth Proportional be b; then $c : t :: y : b$; therefore $b = \frac{ty}{c}$; but (by Prop. 1.) $t^2 : c^2 :: \overline{t - x} \times x : y^2$; therefore (by *Eu.* 22. 6.) $t : c :: \sqrt{\overline{t - x} \times x} : y$, and $\sqrt{\overline{t - x} \times x} = \frac{ty}{c} = b$. Q. E. D.

PROPOSITION VII.

The Diſtance between the Foci, is a mean Proportional between the tranſverſe Axe, and the Difference of the tranſverſe Axe and the Parameter; that is, AB : KH :: KH : AB — LR.

DEMONSTRATION.

Becauſe $\overline{KD}^2 - \overline{CD}^2 = KC^2$; that is, $\frac{1}{4}t^2 - \frac{1}{4}c^2 = \frac{1}{4}b^2$, or $t^2 - c^2 = b^2$; but $pt = c^2$; therefore $t^2 - pt = b^2$, and $t : b :: b : t - p$; or, AB : KH :: KH : AB — LR. Q. E. D.

PROPOSITION VIII.

As the Square of any Ordinate, is to the Rectangle of the Abſciſſas, ſo is the Square of the Conjugate, to the Square of the Conjugate added to the Square of the Diſtance of the Foci; that is, $\overline{FG}^2 : AG \times GB :: \overline{ED}^2 : \overline{ED}^2 + \overline{KH}^2$.

DEMONSTRATION.

Becauſe $\overline{KE}^2 = \overline{KC}^2 + \overline{CE}^2$; that is, $\frac{1}{4}t^2 = \frac{1}{4}b^2 + \frac{1}{4}c^2$; or $t^2 = b^2 + c^2$; but (by Prop. 1.) $y^2 : \overline{t - x} \times x :: c^2 : t^2 = b^2 + c^2$; or, $\overline{FG}^2 : AG \times GB :: \overline{ED}^2 : \overline{ED}^2 + \overline{KH}^2$. Q. E. D.

PROPOSITION IX.

As the Square of any Ordinate, is to the Rectangle of its Abſciſſa into the Parameter, ſo is the Differenee between the Square of the conjugate Axe, and the Rectangle of the Abſciſſa into the Parameter, to the Square of the conjugate Axe; that is, $\overline{FG}^2 : BG \times LR :: ED^2 - BG \times LR : \overline{ED}^2$.

DEMONSTRATION.

From the Equation of the Curve, $t^2 y^2 = c^2 tx - c^2 x^2$; but $t = \frac{c^2}{p}$; therefore, by Subſtitution, $\frac{c^4 y^2}{p^2} = \frac{c^4 x}{p} - c^2 x^2$, and $c^2 y^2 = c^2 px - p^2 x^2$; that is, $y^2 : px :: c^2 - px : c^2$; or, $\overline{FG}^2 : BG \times LR :: \overline{ED}^2 - BG \times LR : \overline{ED}^2$. Q. E. D.

PROPOSITION X.

As the Square of the conjugate Axe, is to the Square of the tranſverſe Axe, ſo is the Rectangle of any two Abſciſſas of the conjugate Axe, to the Square of the Ordinate which divides them; that is, $\overline{DE}^2 : \overline{AB}^2 :: Dh \times Eh : \overline{Fh}^2$.

DEMONSTRATION.

Let $Eh = x$, and $Fh = y$; then (by Prop. 1.) $\overline{AB}^2 : \overline{ED}^2 :: AG \times GB : \overline{FG}^2$; but (by *Eu.* 5. 2.) $AG \times GB = \overline{CB}^2 - \overline{Fh}^2$, and $\overline{Ch}^2 = \overline{FG}^2 = \overline{CE}^2 - Dh \times Eh$; therefore, by Subſtitution, $\overline{AB}^2 : \overline{ED}^2 :: \overline{BC}^2 - \overline{Fh}^2 : \overline{CE}^2 - Dh \times Eh$; that is, $t^2 : c^2 :: \frac{1}{4} t^2 - y^2 : \frac{1}{4} c^2 - cx + x^2$; which reduced to an Equation, produces $c^2 y^2 = t^2 cx - t^2 x^2$; that is, $c^2 : t^2 :: \overline{c - x} \times x : y^2$; or, $\overline{DE}^2 : \overline{AB}^2 :: Dh \times Eh : \overline{Fh}^2$. Q. E. D.

DEFINITION.

A third Proportional to the Conjugate and tranſverſe Axe, is a Parameter to the conjugate Axe; that is, p being put for the Parameter, $c : t :: t : p$; therefore $cp = t^2$.

PROPOSITION XI.

As the conjugate Axe, is to its Parameter, ſo is the Rectangle of any two Abſciſſas of the conjugate Axe, to the Square of the Ordinate which divides them.

DEMONSTRATION.

For t^2, in the laſt Equation, put its Equal cp; then $cy^2 = cpx - px^2$; that is, $c : p :: \overline{c - x} \times x : y^2$. Q. E. D.

PROPOSITION XII.

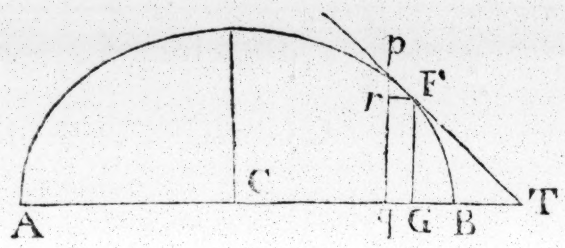
As the Square of any Ordinate of the Conjugate, is to the Rectangle of the Abſciſſas which it divides, ſo is the Sum of the Squares of the Diſtance of the Foci, and the conjugate Axe, to the Square of the conjugate Axe.

DEMONSTRATION.

By the tenth Prop. $y^2 : cx - x^2 :: t^2 : c^2$, and (by *Eu.* 47. 1.) $t^2 = b^2 + c^2$; therefore, by Subſtitution, $y^2 : cx - x^2 :: b^2 + c^2 : c^2$; that is, $\overline{\mathrm{F}b}^2 : \mathrm{D}b \times \mathrm{E}b :: \overline{\mathrm{KH}}^2 + \overline{\mathrm{ED}}^2 : \overline{\mathrm{ED}}^2$. Q. E. D.

PROPOSITION XIII.

In any Tangent to the Ellipſe, if, from the Point of Contact, an Ordinate be drawn to the Axe, and the Tangent continued meet the Axe produced, then it will be, as the Diſtance (in the Axe) between the Center and the Ordinate, is to the Abſciſſa of that Ordinate, ſo is the Remainder of the Axe, to (the Diſtance between the Ordinate and the Interſection of the Tangent with the Axe; that is) the Subtangent, *viz.* CG : GB :: AG : GT.



DEMONSTRATION.

Let FP, an indefinitely ſmall Part of the Curve, be continued to meet the Axe produced in T; draw the Ordinate FG, and parallel to it pq; draw alſo Fr parallel to the Axe, and for Fr put n, pr, m, and BT, a; then is $Bq = x + n$, $Aq = t - x - n$, $pq = y + m$, and $GT = a + x$; but, by ſimilar Triangles, $pr : Fr :: FG : GT$; that is, $m : n :: y : a + x$; therefore $n \times \frac{y}{m} = a + x$, and (by Prop. 2.) $t : p :: tx - x^2 + tn - 2nx - n^2 : y^2 + 2my + m^2$; alſo, $t : p :: tx - x^2 : y^2$; therefore $ptx -$

$px^2 + ptn - 2pnx = ty^2 + 2tmy$, and $ty^2 = ptx - px^2$; conſequently $ptx - px^2 + ptn - 2pnx - 2tmy = ty^2 = ptx - x^2$; or, $ptn - 2npx = 2tmy$, and $n = \frac{2tmy}{pt - 2px}$; therefore $x + a = \frac{n \times y}{m} = \frac{2tmy}{pt - 2px} \times \frac{y}{m} = \frac{ty^2}{p} \times \frac{2}{t - 2x} =$ (becauſe by the ſecond. Prop. $tx - x^2 = \frac{ty^2}{p}$) $\frac{2tx - 2x^2}{t - 2x} = \frac{tx - x^2}{\frac{1}{2}t - x}$; therefore $\frac{1}{2}t - x : x :: t - x : x + a$; or, CG : GB :: AG : GT. Q. E. D.

PROPOSITION XIV.

As the Diſtance from the Center to the Ordinate drawn from the Point of Contact, is to half the tranſverſe Axe, ſo is half the tranſverſe Axe, to the Diſtance from the Center to the Concurring of the Tangent with the Axe produced; that is, CG : CB :: CB : CT.

DEMONSTRATION.

Becauſe, $CT = CG + GT$, and $CT = \frac{1}{2}t + a$, $CG = \frac{1}{2}t - x$; whence (by Prop. 13.) $GT = \frac{tx - x^2}{\frac{1}{2}t - x}$; therefore $\frac{1}{2}t + a = \frac{1}{2}t - x + \frac{tx - x^2}{\frac{1}{2}t - x} = \frac{\frac{1}{4}t^2}{\frac{1}{2}t - x}$; that is, $\frac{1}{2}t - x : \frac{1}{2}t :: \frac{1}{2}t : \frac{1}{2}t + a$; or, CG : CB :: CB : CT. Q. E. D.

PROPOSITION XV.

As the Diſtance from the Center to the Ordinate drawn from the Point of Contact, is to half the Tranſverſe, ſo is the Abſciſſa of that Ordinate, to the external Part of the Tranſverſe; that is, CG : CB :: GB : BT.

DEMONSTRATION.

By Prop. 14. $\frac{1}{2}t+a=\frac{\frac{1}{4}t^2}{\frac{1}{2}t-x}$; therefore $\frac{1}{4}t^2+\frac{1}{2}ta=\frac{1}{2}tx-ax=\frac{1}{4}t^2$, and $a=\frac{\frac{1}{2}tx}{\frac{1}{2}t-x}$; that is, $\frac{1}{2}t-x:\frac{1}{2}t::x:a$; or, CG : CB :: GB : BT. Q. E. D.

PROPOSITION XVI.

As the Diſtance from the Center to the Ordinate drawn from Point of the Contact, is to half the tranſverſe Axe, ſo is the greater Abſciſſa of that Ordinate, to the tranſverſe Axe added to the external Part; that is, CG : CB :: AG : AT.

DEMONSTRATION.

By the 15, $a=\frac{\frac{1}{2}tx}{\frac{1}{2}t-x}$; therefore $t+a=t+\frac{\frac{1}{2}tx}{\frac{1}{2}t-x}=\frac{\frac{1}{2}t^2-\frac{1}{2}tx}{\frac{1}{2}t-x}$; that is, $\frac{1}{2}t-x:\frac{1}{2}t::t-x:t+a$; or, CG : CB :: AG : AT. Q. E. D.

PROPOSITION XVII.

As the greater Abſciſſa of the Ordinate drawn from the Point of Contact, is to the Sum of the Tranſverſe and external Part, ſo is the leſs Abſciſſa of that Ordinate, to the external Part; that is, AG : AT :: BG : BT.

DEMONSTRATION.

By the 15, $\frac{1}{2}t - x : \frac{1}{2}t :: x : a$, and by the 16, $\frac{1}{2}t - x : \frac{1}{2}t :: t - x : t + a$; therefore, by Equality, $t - x : t + a :: x : a$; or, AG : AT :: BG : BT.

PROPOSITION XVIII.

As the Diſtance from the Center to the Concurring of the Tangent, is to half the Tranſverſe, ſo is the external Part, to the Abſciſſa of the Ordinate drawn from the Point of Contact; that is, CT : CB :: BT : BG.

DEMONSTRATION.

By the 15, $\frac{1}{2}ta = \frac{1}{2}tx + xa$; therefore $x = \frac{\frac{1}{2}ta}{\frac{1}{2}t + a}$, and $\frac{1}{2}t + a : \frac{1}{2}t : a : x$; or, CT : CB :: BT : BG.

PROPOSITION XIX.

As half the Tranſverſe added to the external Part, is to the Tranſverſe added to the external

Part, ſo is the external Part, to the Subtangent; that is, CT : AT :: BT : GT.

DEMONSTRATION.

By the 18, $x = \frac{\frac{1}{2}ta}{\frac{1}{2}t + a}$; therefore $x + a = a + \frac{\frac{1}{2}ta}{\frac{1}{2}t + a} = \frac{ta + a^2}{\frac{1}{2}t + a}$, and $\frac{1}{2}t + a : t + a :: a : x + a$; or, CT : AT :: BT : GT. Q. E. D.

PROPOSITION XX.

As the greater Abſciſſa of the Ordinate drawn from the Point of Contact, is to half the Tranſverſe, ſo is the Subtangent, to the external Part; that is, AG : CB :: GT : BT.

DEMONSTRATION.

By the 15, $\frac{1}{2}t - x = \frac{\frac{1}{2}tx}{a}$; therefore $t - x = \frac{1}{2}t + \frac{\frac{1}{2}tx}{a} = \frac{\frac{1}{2}ta + \frac{1}{2}tx}{a}$, and $t - x : \frac{1}{2}t :: x + a : a$; or, AG : CB :: GT : BT. Q. E. D.

PROPOSITION XXI.

As the Tranſverſe added to the external Part, is to half the Tranſverſe, ſo is the Subtangent, to the Abſciſſa; that is, AT : CB :: GT : GB.

DEMONSTRATION.

By the 18, $\frac{1}{2}t + a = \frac{\frac{1}{2}ta}{x}$; therefore $t + a = \frac{1}{2}t + \frac{\frac{1}{2}ta}{x} = \frac{\frac{1}{2}tx + \frac{1}{2}ta}{x}$, and $t + a : \frac{1}{2}t :: x + a : x$; or, AT : CB :: GT : GB. Q. E. D.

PROPOSITION XXII.

The Ordinate drawn from the Point of Contact, divided by the Subtangent, is equal to the Quotient of the Diſtance between the Center and that Ordinate divided by that Ordinate, multiplied by the Parameter divided by the tranſverſe Axe; that is, $\frac{GF}{GT} = \frac{CG}{GF} \times \frac{p}{t}$.

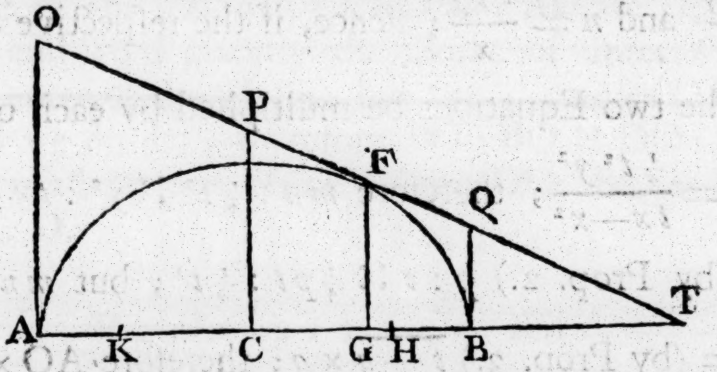
DEMONSTRATION.

By the 13, $tx - x^2 = \overline{\frac{1}{2}t - x} \times \overline{x + a}$; and (by Prop. 2.) $t : p :: tx - x^2 : y^2 :: \overline{\frac{1}{2}t - x} \times \overline{x + a} : y^2$; therefore $ty^2 = p \times \frac{1}{2}\overline{t - x} \times \overline{x + a}$; which being divided by $\overline{x + a} \times ty$, it produces $\frac{y}{x + a} = \frac{\overline{\frac{1}{2}t - x}}{y} \times \frac{p}{t}$. Q. E. D.

PROPOSITION XXIII.

If Perpendiculars be drawn from the Extremities of the Tranſverſe, and from the Center, meeting any Tangent, and alſo if from the Point of

Contact be drawn an Ordinate, theſe four Lines will be proportional; that is, AO : CP :: FG : BQ.



DEMONSTRATION.

By the 19, TA : TC :: TG : TB; therefore (by *Eu*. 4. 6.) AO : CP :: FG : BQ. Q. E. D.

COROLLARY.

AO × BQ = CP × FG.

PROPOSITION XXIV.

If Perpendiculars be drawn from the Extremities of the Tranſverſe, meeting any Tangent; then the Rectangle of theſe Perpendiculars, will be equal to the Rectangle of the greateſt and leaſt Diſtance of either of the Foci from the Vertices; that is, AO × BQ = AH × BH = BK × AK.

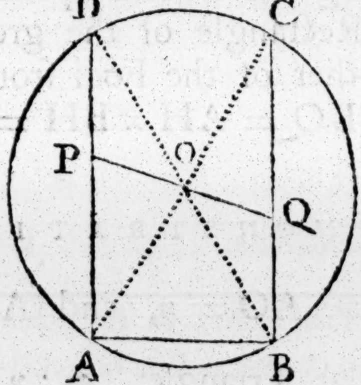
DEMONSTRATION.

Let BQ $= m$, AO $= n$, and AK = BH $= q$; then, by ſimilar Triangles, $m : y :: a : a + x ::$

(by Prop. 20.) $\frac{1}{2}t : t - x$; and $n : y :: t + a$ $a + x ::$ (by Prop. 21.) $\frac{1}{2}t : x$; therefore $m = \frac{\frac{1}{2}ty}{t - x}$ and $n = \frac{\frac{1}{2}ty}{x}$; hence, if the reſpective Sides of the two Equations be multiplied by each other, $mn = \frac{\frac{1}{4}t^2y^2}{tx - x^2}$; therefore $mn : \frac{1}{4}t^2 :: y^2 : tx - x^2 ::$ (by Prop. 2.) $p : t :: \frac{1}{4}pt : \frac{1}{4}t^2$; but $mn = \frac{1}{4}pt =$ (by Prop. 3.) $\overline{t - q} \times q$; therefore AO × BQ = AH × BH = BK × AK. Q. E. D.

LEMMA.

If on the Extremities of any Subtenſe of a Circle, as AB, the Perpendiculars AD, BC be erected meeting the Periphery in the Points D, C; and from theſe Points to the oppoſite Extremities, B and A, of that Subtenſe, be drawn two right Lines DB and CA, they will interſect each other in (O) the Circle's Center; through which, alſo, if a right Line be drawn any how, it will make the alternate Segments of the Perpendiculars equal.

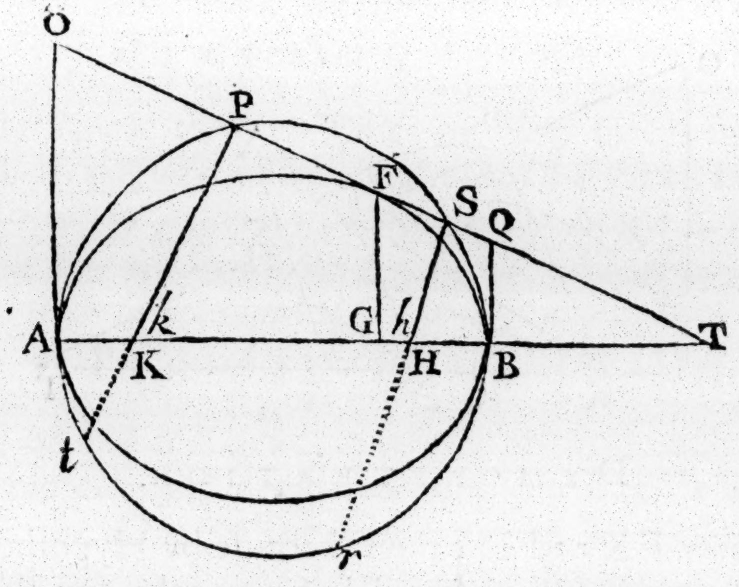


DEMONSTRATION.

By Hypotheſis the Angles B and A are right; therefore (by *Eu.* 31. 3.) BD and AC are Diameters, and conſequently the Point of Interſection the Center of the Circle; but the Triangles OPD, OQB are ſimilar; therefore BO : BQ :: DO : DP, and conſequently, ſince BO = DO, BQ = DP. Q. E. D.

PROPOSITION XXV.

If from the Interſections (P, S) of a Circle, whoſe Diameter is the tranſverſe Axe, with any Tangent, Perpendiculars P*k*, S*h* be drawn, they will cut the tranſverſe Axe in the focal Points; that is, the Points *k*, *h* coincide with K, H.



DEMONSTRATION.

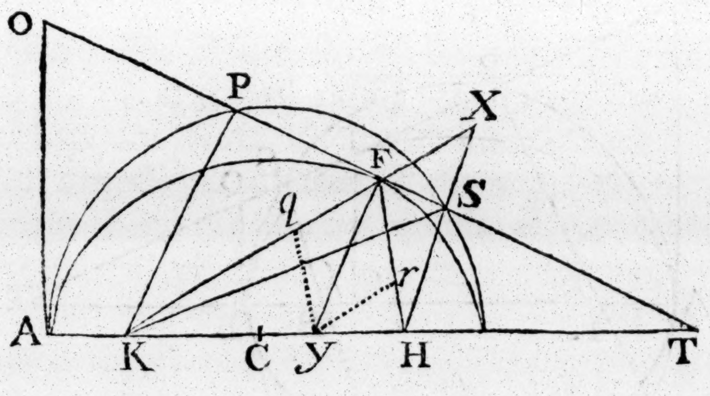
The Triangles TBQ, ATO are ſimilar to the Triangles TS*h*, TP*k*, each having a right Angle, and the Angle T common; therefore $AO : Pk :: Sh : BQ$, and $AO \times BQ = Pk \times Sh =$ (by the precedent *Lemma*) $Pk \times kt$; or, $hr \times Sh =$ (by *Eu.* 35. 3.) $Ak \times Bk$, or $Bh \times Ah$; but $AO \times BQ = Ak \times BK$ or $AH \times HB$ (by Prop. 24.); therefore the Points H, *h* and K, *k* are coincident. Q. E. D.

COROLLARY.

It is manifeſt that, $KP \times HS = \frac{1}{4} pt$; becauſe $KP \times HS = AK \times BK = \overline{t-q} \times q =$ (by Prop. 3.) $\frac{1}{4} pt$.

PROPOSITION XXVI.

If to any Point of the Curve right Lines be drawn from the Foci, and one of the Lines be continued; then a right Line biſecting the external Angle, will touch the Curve in the angular Point.



DEMONSTRATION.

Take $FX = FH$; then (becauſe by Hypotheſis, the Angle $TFX = HFT$) if you take any Point S, in the Line FT; $HS = SX$ (by *Eu.* 4. 1.). Draw KS; then $KS + SX = KS + HS$ is greater than KX; or its Equal AB, and therefore the Point S, is without the Curve; for if it were in the Curve $KS + HS$ (by the Geneſis) $= AB$.

PRO-

PROPOSITION XXVII.

Lines drawn from the Foci to the Point of Contact, make equal Angles with the Tangent.

DEMONSTRATION.

By the precedent Prop. the Angle HFT = TFX = (by *Eu.* 15. 1.) KFO. Q. E. D.

PROPOSITION XXVIII.

A right Line perpendicular to the Tangent at the Point of Contact, biſects the Angle formed by Lines drawn from the Foci to the ſame Point; that is, if FY be perpendicular to OT; then the Angle KFY = HFY.

DEMONSTRATION.

The Angle PFY = TFY, by Hypotheſis, from which if there be taken the Angle KFP = HFS (by Prop. 27.) there will remain the Angle KFY = HFY. Q. E. D.

PROPOSITION XXIX.

If, on the Tangent, at the Point of Contact, a Perpendicular be drawn meeting the Axe, it will divide the Diſtance between the Foci, in the ſame Proportion, as Lines drawn from the Foci to the ſame Point; that is, FK : FH ∷ KY : HY.

DEMONSTRATION.

In the Triangle HFK, the Angle KFY = HFY (by Prop. 28.); therefore (by *Eu.* 3. 6.) FK : FH :: KY : HY. Q. E. D.

PROPOSITION XXX.

If, on the Tangent, at the Point of Contact, a Perpendicular be drawn, and if, from the Point where that Perpendicular meets the Axe, Lines be drawn perpendicular, to Lines drawn from the Foci to the Point of Contact; then the Diſtance on theſe Lines, from the Point of Contact, to the Perpendiculars, will be equal to half the Parameter of the Axe; that is, $Fq = Fr = \frac{1}{2}p$.

DEMONSTRATION.

From the Points S, P, where a Circle on the Tranſverſe cuts the Tangent, draw the Lines SH, PK to the Foci, which will be perpendicular to PT, by 25. and conſequently parallel to FY; continue KF, HS, till they concur in X; then $KX = t$, and $HX = 2HS$ by the 26. and becauſe the Triangles KFY, KPF are reſpectively ſimilar to the Triangles KXH, qFY; therefore $KX : HX :: FK : FY :: KP : Fq$, and $KX \times Fq = HX \times KP$; but $\frac{1}{2} HK \times Fq = \frac{1}{2} HX \times KP$; that is, $\frac{1}{2} t \times Fq = HS \times KP =$ (by Prop. 25.) $\frac{1}{4} pt$; therefore $Fq = \frac{1}{2} p$; but (by Prop. 28.) the Angle $YFq = YFr$, and (by *Eu.* 26. 1.) $Fr = Fq$; therefore $Fq = Fr = \frac{1}{2} p$. Q. E. D.

To be continued.

ANSWERS
TO THE
PROBLEMS
PROPOSED IN
NUMBER I.

PROBLEM I. *Anſwered by Mr.* Thomas Hulme *of* London.

LET x repreſent the Number of Perſons that agreed with the Waterman; then $\frac{x}{4}+3$ will be the Number of Perſons taken in by the Way, and $\frac{3x}{2}+18$ the Pence they paid; half of which divided by x, will be $\frac{3}{4}+\frac{9}{x}$ what each of the firſt Perſons gained, by Means of thoſe taken in afterwards, which by the Conditions of the Problem $=1$; therefore $x=\frac{9}{1-\frac{3}{4}}=36$.

PROBLEM II. *Anſwered.* R.

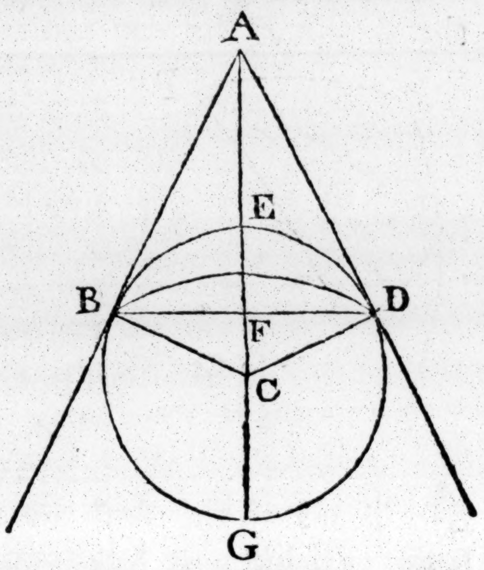
DEFINITION.

The Weight of a Body compared with that of another Body of equal Magnitude, is called its ſpecific Gravity.

Now the ſpecific Gravities of Bodies may be thus determined by Experiment: Let the Body whoſe ſpecific Gravity is required, be firſt weighed in Air, afterwards in Water; then the ſpecific Gravity of the Body, is to that of Water, as the Weight of the Body in Air, to the Weight loſt in Water: For *(by the Definition)* the Weight of Water of the ſame Magnitude with the Body, is to the Weight of the Body, as the ſpecific Gravity of Water, to the ſpecific Gravity of the Body: Hence it follows, if the Water be ſpecifically heavier than the Body; that the ſpecific Gravity of the Body, is to that of Water, as the immerged Part of the Body, to the Magnitude of the Body: But the ſpecific Gravity is as the Denſity; therefore it may very juſtly be as Mr. *Turner* expreſſes it: As the Denſity of Water, to the Denſity of the Body, ſo is the Magnitude of the Body, to the Magnitude of the Space which the Body poſſeſſes in the Water, and ſo is one Foot, the Side of the Cube, to the Part thereof immerged: Therefore let a $(= 1728 \times 0.527458$, Ounces-Troy$)$ be the Weight of a Cubic-Foot of Water, and b $(= 1728 \times 0.489008$ ſuch Ounces$)$ that of Oak; then $a - p$ $(= 66.4416 \times \frac{1}{12} = 5.5368$ Pounds$)$ the Weight to be laid on, and $a : b :: 1 : \frac{a}{b}$, the Part thereof immerged; consequently $1 - \frac{a}{b}$ $\left(= 1 - \frac{0.489008}{0.527458}\right.$, or $12 - \frac{0.489008 \times 12}{0.527458}$ Inches$)$ the Thickneſs of the Part above the Water.

PROBLEM III. *Anſwered by Mr.* John Turner *of* London.

Since the Curve-Surface of any Fruſtum of a Sphere, is to the whole Surface, as the Height of the Fruſtum to the Diameter of the Sphere; EF will be to EG as 2 to 5; therefore EF will be to EC as 2 to $2\frac{1}{2}$, or as 4 to 5, and FC to EC as 1 to 5; but by ſimilar Triangles, CF : BC :: BC : AC; that is, 1 : 5 :: B : 5 BC = AC; therefore AE = 4 BC, or twice the Diameter.

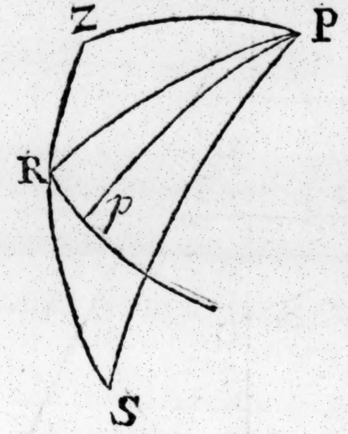


The ſame anſwered. R.

If GBED, the Section of the Earth, with the Plane of the Meridian be a Circle, and Lines be drawn to the ſame according to the Import of the Problem, as in the Figure: It will appear that, GF × FE = CF × FA, or by the Property of the Sphere, mentioned in the foregoing Solution, $\frac{3\,GE \times 2\,GE}{25} = \frac{GE \times FA}{10}$; therefore FA = $\frac{12\,GE}{5}$, and conſequently AE = 2 GE, the ſame as above.

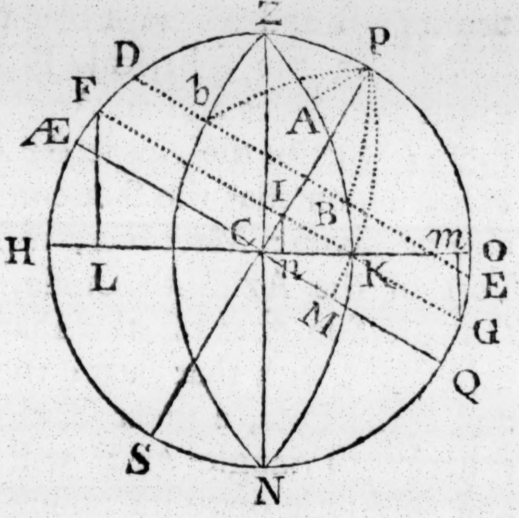
PROBLEM IV. *Anſwered by Mr.* John Turner *of* London.

Let ZP be the Complement of Latitude, and the Angle Z the Azimuth of the Plane; alſo, let S repreſent the Place of *Sirius* when in the Plane, *p* that of *Pollux* at the ſame Time, and R*p* a Part of the Parallel of Declination deſcribed by *Pollux* in $(56'\ 20'')$ the given Time: Then, ſince the Angle RP*p* anſwering to the given Time, and *p*PS that anſwering to the Difference of Right-Aſcenſions of the two Stars are given, their Sum RPS will likewiſe be given; therefore in the Triangle SPR will be given two Sides and the Angle included, whence the Angle S will be known; then in the Triangle ZPS will be given two Sides and an Angle, from whence the Angle Z will alſo be known, whoſe Difference from a right Angle is the Declination required; which (ſuppoſing the Right-Aſcenſion of *Pollux* $= 112^\circ\ 12'$, the Declination North $28^\circ\ 51'$, the Right-Aſcenſion of *Sirius* $98^\circ\ 29'$, the Declination South $= 16^\circ\ 17'$) will be $= 54^\circ\ 44'$.



PROBLEM V. *Anſwered.* R.

Let HZON be the Meridian, HO the Horizon, ÆQ the Equator, making an Angle with the ſame equal to the Complement of Latitude, NZ the prime Vertical, PS the Axis of the Sphere and Hour Circle of Six, in the required Latitude OP, NKZ an Azimuth Circle, upon the Plane of which the erect declining Dial is ſuppoſed to be drawn; alſo let PA be perpendicular to that Azimuth Circle: Then in the right-angled ſpherical Triangle AZP, it is manifeſt, that PZ may repreſent the Complement of the Latitude, AP the Height of the Style, AZ the Diſtance of the Subſtyle from the Meridian, the Angle AZP the Complement of the Plane's Declination, and the Angle APZ the Inclination of the Meridians: Therefore, ſince AP, AZ and the Angle AZC, are equal to each other, by the State of the Problem, let the Sine of any one of them be put $= x$; then the Tangent of AZ $= \frac{x}{\sqrt{1 - x^2}}$, and that of the Angle AZP $= \frac{\sqrt{1 - x^2}}{x}$; whence (*by ſpherical Trigonom.*) $x : 1 :: \frac{x}{\sqrt{1 - x^2}} : \frac{\sqrt{1 - x^2}}{x}$; therefore $\sqrt{1 - x^2} = \frac{x}{\sqrt{1 - x^2}}$, and conſequently $x^2 + x = 1$, or $x = \frac{\sqrt{5} - 1}{2} = 0.61803$, &c. the Sine of (38° 10′) the Plane's Declination; whence the Latitude = 38° 10′.



But to find what Time the Sun comes on and goes off the Plane: Let EB*b*D be the Parallel of Declination deſcribed by the Sun on the Day propoſed; then it is manifeſt, that the Sun firſt comes on the Plane at B, where it is coincident with the

Azimuth Circle, already defined; therefore, if through B and P the Pole a great Circle be deſcribed, in the ſpherical Triangle BZP will be given two Sides and an Angle oppoſite to one of them; whence the Angle BPZ, ſhewing the Time before Noon that the Sun comes on the Plane, will alſo be given: Moreover, if N*b*Z be the farther Part of the Azimuth Circle, it is alſo manifeſt, that the Sun goes off the Plane at *b*, where it is coincident with that Part; therefore, if through *b* and P the Pole, a great Circle be deſcribed, in the ſpherical Triangle *b*ZP will be given two Sides and an Angle oppoſite to one of them, as in the former Caſe; whence the Angle *b*PZ, ſhewing the Time paſt Noon that the Sun goes off the Plane, will alſo be given.

But if it were alſo required, to find the Declination deſcribed by the Sun, when it continues the longeſt on the Plane: Through K, the Interſection of the Horizontal and Azimuth Circle let there be drawn FG, and alſo through K and P the Pole, let a great Circle be deſcribed, meeting the Equator in M; then in the right-angled ſpherical Triangle KCM will be given CK, the Plane's Declination, and the Angle KCM the Complement of Latitude; whence KM, the Sun's Declination, will likewiſe be given.

PROBLEM VI. *Anſwered by Mr.* John Turner *of* London.

Let HPOS, in the former Figure, be an Orthographick Projection of the Sphere; in which HO repreſent the Horizon, ÆQ the Equator, FG the Parallel of Declination, PS the Axis of the Sphere; alſo, let I*n* repreſent the Sun's Altitude at Six, and let the Perpendiculars FL and G*m* be drawn: Then, by the Property of the El-

Ellipſis, and ſimilar Triangles, it will be, as $CM : ÆM :: IK : FK :: In : FL$; and $CM : MQ :: IK : KG :: In : mG$; whence the Arches FH ($50° 45' 44''$) and GO ($14° 52' 2''$) become known: Thus far Mr. *Turner*: But in order to obtain the Declination and Latitude from thence, it is manifeſt that the Arch $HF - ÆF - GO = ÆF$; therefore $ÆF = \frac{HF - GO}{2}$, and conſequently $HÆ = \frac{HF - GO}{2} + GO = \frac{HF + GO}{2}$: Hence proceed the two following Theorems, generally made uſe of, for that Purpoſe.

THEOREM I.

From the Meridian Altitude, ſubſtract the Depreſſion at Midnight, and the Half of that Remainder, will be equal to the Declination.

THEOREM II.

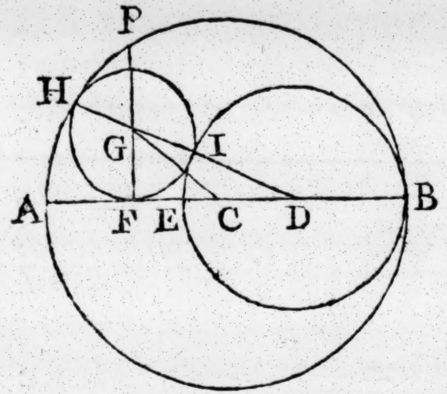
To the Meridian Altitude, add the Depreſſion at Midnight, and the Half of that Sum, will be equal to the Complement of the Latitude.

Whence the Latitude, in the preſent Caſe, is $56° 41' 7''$ and the Declination $18° 26' 51''$; which anſwers to the ſecond Day of *May*. R.

PROBLEM VII. *Anſwered by Mr.* John Turner *of* London.

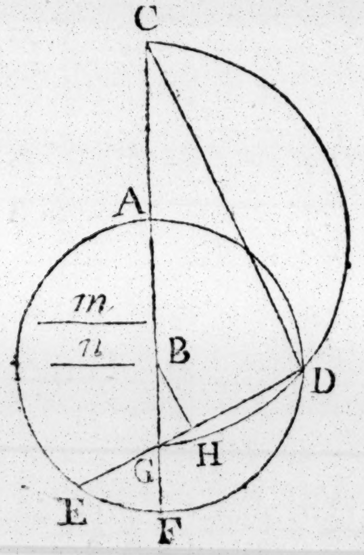
Join the Centers of the given Circles, and on AB let fall the Perpendicular GF; putting $BD = a$, $BC = b$, $BF = x$, and $FG = y$; then (by 47. *Eu.* 1.)

we ſhall have $\overline{x-a}|^2+y^2=\overline{a+y}|^2$ and $\overline{x-b}|^2+y^2=\overline{b-y}|^2$: From the former of which $y=\frac{x^2-2ax}{2a}$, and from the latter $y=\frac{2bx-x^2}{2b}$; therefore $\frac{x^2-2ax}{2a}=\frac{2bx-x^2}{2b}$, and conſequently $x=\frac{4ab}{a+b}$: Whence this Conſtruction: Take BF, a fourth Proportional to BD+BC, BE and BA, and make FP perpendicular to BA; in which take FG a fourth Proportional to BE, BF and FE; then will G be the Center, and FG the Radius of the Circle to be deſcribed.



PROBLEM VIII. *Anſwered by the ſame.*

Let AEFD be the given Circle; B the Center thereof, and G the given Point; and let the Lines *m*, *n* be in the given Ratio of the Parts DG, EG: Take BC to BG as $m+n$ to $m-n$, and upon the Diameter GC let a Circle be deſcribed, cutting the former in D; draw DGE and the Thing is done.



De-

DEMONSTRATION.

Draw CD, and BH parallel thereto; then $m+n : m-n$, or $\frac{m+n}{2} : \frac{m-n}{2} :: CB : BG :: DH : GH$; therefore by Compoſition and Diviſion $m : n :: DG : EG$. Q. E. D.

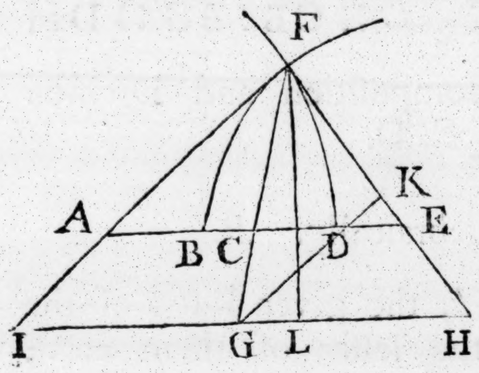
Otherwiſe Algebraically. *R.*

Let $GF = a$, $AG = b$, the given Ratio of the Parts as m to n, and the leſſer of them $= x$; then the greater will be $\frac{mx}{n}$; whence, by the Property of the Circle, $\frac{mx^2}{n} = ab$, and conſequently $x = \sqrt{\frac{nab}{m}}$: From whence proceeds the following Conſtruction, given by *Tycho Oxonienſis*. Let the given Point be G, and the given Ratio of the Parts as R to S: Then take the Line $m : FG :: S : R$, and between AG and m, find a mean Proportional EG, which apply in the given Circle from G to the Periphery; continue EG till it cuts the Circle in D, and DG will be to $EG :: R : S$. For (by Conſtruction) $AG : EG :: EG : m$, and $AG : EG :: DG : FG$ (by 35. *Eu.* 3.); therefore $DG : FG :: EG : m$, and, by Permutation, $DG : EG :: FG : m$; that is, as R to S. Q. E. D.

PROBLEM IX. *Anſwered by Mr.* John Turner.

CONSTRUCTION.

Draw AE at Pleaſure, which biſect at C, and take AB to BC in the Ratio of one Side to the biſecting Line, and CD to DE in the Ratio of the biſecting Line to the other Side, then according to the Method laid down in Lemma Page 310 of Mr. *Simpſon*'s Algebra, let two Circles be deſcribed cutting each other in F; draw FA, FE and FC, in which produced, if need be, take FG equal to the biſecting Line, and draw IGH parallel to AE, interſecting FA and FE produced in I and H; then FIH will be the Triangle that was to be conſtructed. The Demonſtration of which is manifeſt from the before mentioned Lemma.



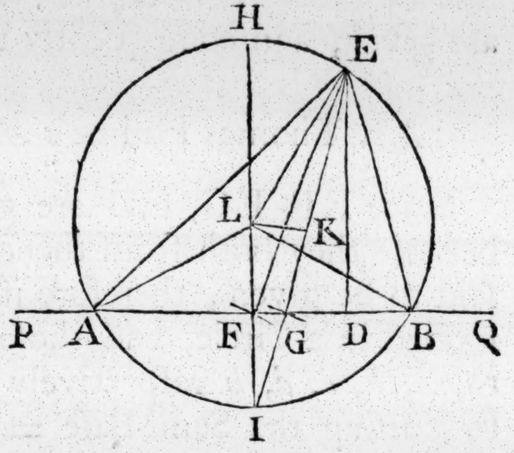
The ſame anſwered. N.

Let FG be the given Line biſecting the Baſe, FI and FH the two given Sides of the Triangle: Then if GK be drawn parallel to FI, it will manifeſtly biſect FH, and be equal to the Half of FI; therefore in the Triangle FGK there will be given all the Sides, from whence the Angle GKH will be given, as well as the Angle GKF; then in the Triangle GKH will be given two Sides, and an Angle included; whence GH will be given, and conſequently the Baſe of the Triangle.

PROBLEM X. *Anſwered by Mr.* John Turner.

CONSTRUCTION.

From any Point D, in the indefinite Line PQ, draw DE perpendicular to PQ and equal to the given Perpendicular; then upon E with Radii reſpectively equal to the two given biſecting Lines, deſcribe two Arcs cutting PQ in F and G; draw HFI perpendicular to PQ, alſo draw EG which produce to meet HI in I; biſect EI with the Perpendicular KL, meeting HI in L, then upon L with LI as Radius, deſcribe the Circle AHBI cutting PQ in A and B; join A, E and B, E, then ABE will be the Triangle that was to be conſtructed.



DEMONSTRATION.

Join E, F; L, E; L, B and L, A: Becauſe LK is perpendicular to and biſects IE, it is evident the Circle paſſes through the Point E. Moreover, becauſe FL is perpendicular to AB, and LA equal to LB, AF will be equal to FB; wherefore it is evident that FE biſects the Baſe, and that the Arches IA, IB as well as the Angles AEI, and BEI are equal. Q. E. D.

Method of Calculation.

As the Line biſecting the vertical Angle, is to the Perpendicular, ſo is Radius, to the Coſine of half the Difference of the Angles at the Baſe: And as the Line biſecting the Baſe, is to the Perpendicular, ſo is Radius, to the Coſine of an Angle, which ſubſtract from the Difference of the Angles at the Baſe found by the preceding Proportion; then ſay, as the Sine of the ſaid Angle, is to the

Sine of the Remainder, ſo is Radius, to the Coſine of the vertical Angle; whence all the Angles are given, and conſequently the Sides.

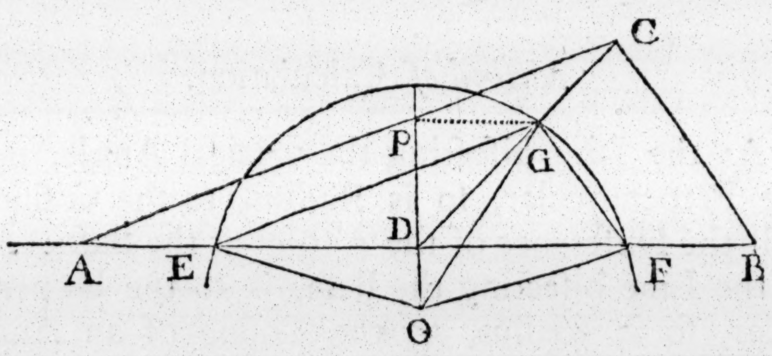
Mr. Thomas Hulme'*s Anſwer to the ſame.*

Since EF, EG, ED are given; FG and GD, becauſe they both from thence may be found, are ſaid to be given; therefore if the Triangle be inſcribed in a Circle, and thoſe Lines be put equal to, a, b, c, d, h reſpectively; we ſhall then have, by putting the Semi-Baſe $= x$; $AG = x + d$ and $GB = x - d$; whence, by ſimilar Triangles, GD : EG :: FG : GI $= \frac{bd}{h}$, and, by the Property of the Circle, $AG \times BG = GI \times EG$; therefore $x^2 - d^2 = \frac{db^2}{h}$, and conſequently $x = \sqrt{\frac{db^2}{h} + d^2}$.

Problem XI. *Anſwered by Mr.* John Turner.

Construction.

Draw AB at Pleaſure, in which take $ED = DF$, and upon EF let a Segment of a Circle be deſcribed to contain an Angle, equal to the given Angle at the Vertex; make BDG equal to the Angle which the biſecting Line makes with the Baſe, and produce DG, if need be, ſo that DC may be



equal to the biſecting Line; join E, G and F, G and draw CA and CB parallel to EG and FG reſpectively, then ABC will be the Triangle required.

DEMONSTRATION.

Becauſe AC and BC, are parallel to EG and GF; the Angles ACD and BCD will be equal to EGD and FGD reſpectively; therefore ACD + BCD = EGD + FGD = EGF = the given Angle at the Vertex by Conſtruction: Alſo, becauſe of the ſimilar Triangles, ACD, EGD and BCD, FGD, we ſhall have GD : DE (DF) ∷ GC : AE = FB; therefore AD = DB. Q. E. D.

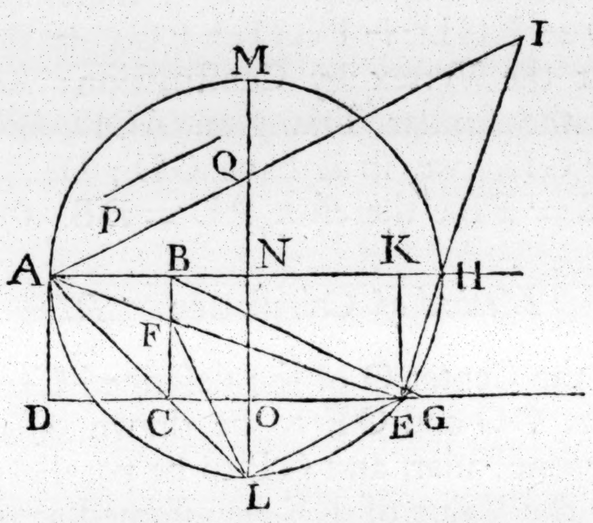
Method of Calculation. *R.*

From the Center O of the deſcribed Circle, conceive OE, OG and OF to be drawn, alſo ODP to be perpendicular, and GP to be parallel to EF: Then if the Value of EF be aſſumed at pleaſure, there will be given in the right-angled Triangle ODF all the Angles, and the Side DF; whence the other two Sides OD, and OF = OG will alſo be given; and therefore, ſince the Angle GDO is given; in the Triangle ODG there will be given, the two Sides OD, OG and an Angle oppoſite to one of them; whence the Angle DOG, the Difference of the Angles at the Baſe, will likewiſe be given; but their Sum is given, becauſe the Angle at the Vertex is given; therefore the Angles themſelves from hence become known: Wherefore in the Triangle ACB are given, the Angles at the Baſe, the Angles which the biſecting Line makes with the Baſe, and the Line itſelf; whence the Sides will alſo be given.

PROBLEM XII. *Anſwered by Mr.* John Turner.

CONSTRUCTION.

Make CG equal to the given intercepted Line, and join B, G; in AB produced take $BH = BG$, and upon the Diameter AH let a Circle be deſcribed cutting DG in E, draw AE and the Thing is done.



DEMONSTRATION.

Draw EK and the Diameter ML perpendicular to DG; alſo, draw EHI, ſo, that EI may be equal to EA, and join A, I; A, L, and L, E: Then by the Property of the Circle $LE^2 = LM \times LO$, therefore $2LE^2 = LM \times 2LO$; but $LM = AH$, is $= BG + AB$ by Couſtruction, and $2LO = 2NL - 2NO = ML - 2NO = ML - 2AB = BG - AB$; conſequently $2LE^2 = \overline{BG + AB} \times \overline{BG - AB} =$ (by *Eu.* 2. 5.) $BG^2 - AB^2 = BG^2 - BC^2 = CG^2$: Therefore, becauſe of the ſimilar Triangles ALE, AHI, it will be as $2AL^2 : 2LE^2\ (CG^2) :: AH^2 : HI^2$; but the Antecedents being equal, the Conſequents will likewiſe be equal, that is $CG^2 = HI^2$, or $CG = HI$.

HI. Moreover, becauſe the equiangular Triangles ABF, EHK have the homologous Sides AB and EK equal to each other, the Side AF will alſo be equal to the Side EH, which being taken from the equal Quantities AE and EI, will leave FE = HI = CG. Q. E. D.

Method of Calculation.

In the Triangle LOE are given the two Sides LO, LE and the right Angle LOE; whence the Side OE will be given, and conſequently the Point E in the Side of the given Square produced.

The ſame anſwered by Tycho Oxonienſis.

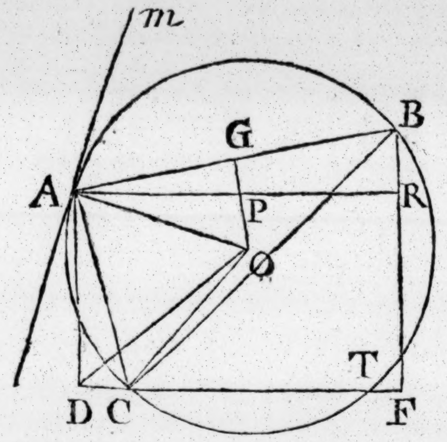
Let CG be the given Line: Take PQ : CG :: DC : AC, and AL : PQ :: PQ : CL; from the Point L apply the Line FL = PQ, and through the Points A and F draw AE; then EF is the Line required.

For ſince by Conſtruction, AL : PQ :: PQ : CL; that is, AL : FL :: FL : CL, the Triangles ALF and CLF, will be equiangular, and the Angle AFL = FCL = ACE; but the Angle LAF is common to both the Triangles AFL, ACE; therefore AC : CE :: AF : FL, that is to PQ; and by Reaſon of the Parallels AD, and CF; DC : CE :: AF : FE; therefore by Equality DC : AC :: PQ : FE; but, by Conſtruction, DC : AC :: PQ : CG; therefore CG = FE. Q. E. D.

PROBLEM XIII. *Anſwered by* Tycho Oxonienſis.

CONSTRUCTION.

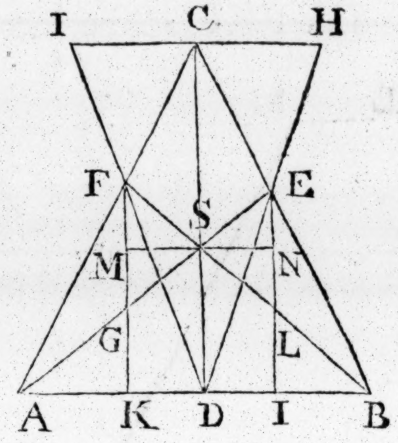
At the Point A, the Extremity of the Line AB, joining the two given Points, make the Angle BA*m* equal to the given Angle D; alſo make AO perpendicular to A*m*, and upon G the Middle of AB erect another Perpendicular, meeting the former in O; on the Point O as a Center, with the Radius AO, deſcribe the Circle ABCT, cutting the given right Line DF in C; join AC, BC and theſe Lines ſhall comprehend the given Angle D. The Demonſtration of which is very obvious, from the 20 and 32. *Eu.* 3.



Method of Calculation.

Draw AR parallel, and AD, BF perpendicular to DF; then in the Triangle BAR will be given AB, BR (= BF — AD) and the right Angle ARB; whence the other two Angles ABR, BAR and the Side AR will be given; but the Angle GAO is given, (from the Conſtruction) and AG = $\frac{AB}{2}$; therefore the Radius AO, and conſequently the Angle PAO will be given; the Difference between which Angle and a right one, is equal to the Angle DAO; therefore in the Triangle DAO are given two Sides and an Angle included, whence OD, the Angle ADO and conſequently the Angle ODC will likewiſe be given; then in the Triangle ODC will be given the two Sides OD, OC (= AO) and the Angle ODC, whence DC will be given, and conſequently the Lines AC, CB containing the given Angle.

PROBLEM XIV. *Anſwered by Mr.* John Turner.

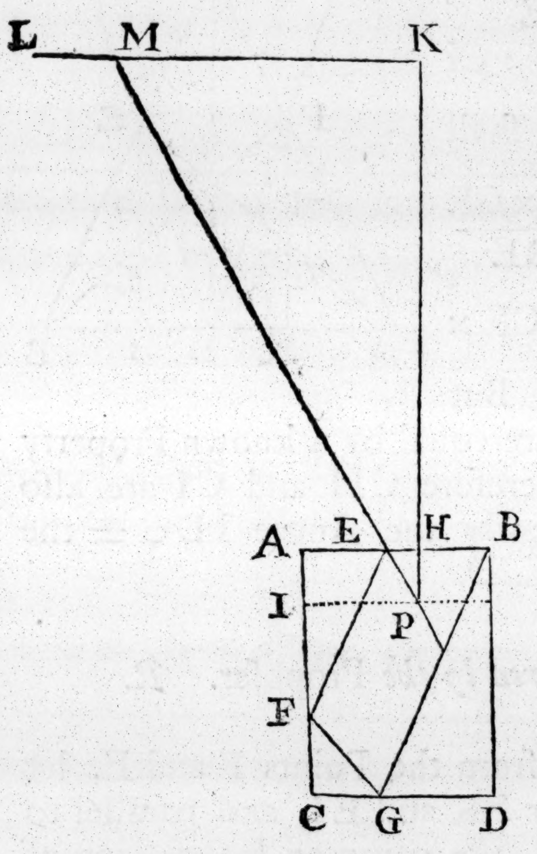


Draw ICH parallel to AB, interſecting DE and DF produced in H and I: Then ſince, by ſimilar Triangles, BE : BD :: CE : CH $= \frac{BD \times CE}{BE}$, and AF : AD :: CF : CI $= \frac{AD \times CF}{AF}$; we ſhall have CH : CI :: $\frac{BD \times CE}{BE} : \frac{AD \times CF}{AF}$ :: BD × CE × AF : AD × CF × BE. But the two laſt Terms are equal by a known Property of the Triangle; therefore CH and CI are alſo equal, and conſequently the Angle IDC = the Angle HDC. Q. E. D.

The ſame anſwered by the Propoſer. *R.*

To the Baſe AB, from the Points F and E, let fall the Perpendicular FK and EI, and parallel to the ſame through S, the common Interſection of the Lines AE, BF and CD, draw MN: Then, by ſimilar Triangles, AC : AF :: CS : GF :: CD : FK; therefore, by Permutation, CS : CD :: GF : FK; after the very ſame Manner, it will appear that CS : CD :: EL : EI; whence, by Equality, GF : FK :: EL : EI, or again, by Permutation, GF : EL :: FK : EI: But GF : EL :: FS : LS :: MS : NS; whence again, by Equality, FK : EI :: MS : NS, or as KD : DI; therefore the two Triangles DFK, DEI, having one Angle K equal to one Angle I, and the Sides about the other Angles DFK, DEI proportional, are equiangular (*Eu.* 7. 6.); and conſequently the Angle FDK, equal to the Angle IDE. Q. E. D.

PROBLEM XV. *Anſwered by Mr.* John Turner.

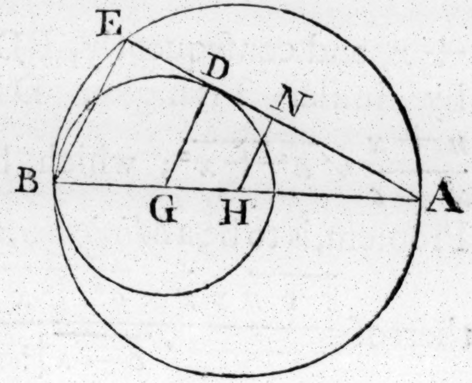


Let PEFGB be the Path of the Ball, and make PH perpendicular to AB, and PI to AC: Then, becauſe the Angles of Incidence and the Angles of Reflection are equal, and the Angles at H, A, C, and D right Angles, the Triangles PHE, FEA, FCG and BGD are all ſimilar; but the Sum of all the Perpendiculars $PH + AF + FC + BD$ is given $= PH + 2BD$, and the Sum of the Baſes $EH + AE + CG + DG$ is alſo given $= AH + CD$; therefore it will be as $PH + 2BD : AH + CD :: PH : HE$; whence this

CONSTRUCTION.

In PH produced take $HK = 2BD$; and in KL perpendicular to PK, take $KM = AH + CD$, and draw PM for the Direction required.

PROBLEM XVI. *Anſwered by Mr.* Thomas Leigh.

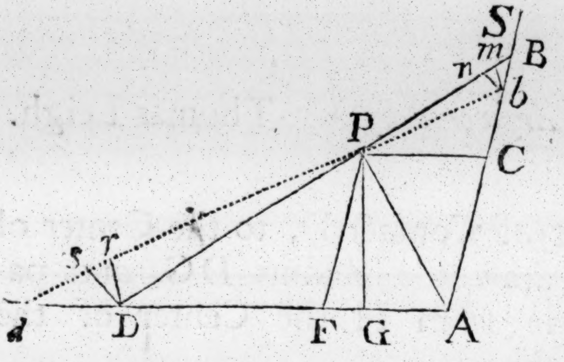


From the Point of Contact D, to the Center of the leſſer Circle, draw the Radius DG, and parallel to the ſame from H the Center of the greater, draw alſo the right Line HN: Then by putting GH $=a$, ED $=b$, and BH, the Radius of the greater Circle, $=x$; we ſhall have AG $=a+x$, BG $=$ GD $=x-a$; whence, (by *Euclid* 36. 3.) DA $=2\sqrt{ax}$; but, by ſimilar Triangles, AB : AG :: AE : AD; therefore, by Diviſion, AB — AG : AG :: AE — AD : AD; that is, BG : AG :: ED : AD, or $x-a : a+x :: b : 2\sqrt{ax}$; therefore $\overline{x-a}|^2\times 4ax=\overline{a+x}|^2\times b^2$; which Equation, by Reduction, becomes $x^3-\overline{2a+\frac{b^2}{4a}}\times x^2+\overline{a^2-\frac{b^2}{2}}\times x=\frac{b^2a}{4}$, from whence the Value of x may be determined, and conſequently the Diameter of the leſſer Circle.

PROBLEM XVII. *Anſwered by Mr.* Thomas Leigh.

Let A*d* and AS be the two right Lines given by Poſition, P the given Point, and DPB the Line required: Then if PF be drawn parallel to

AB,



AB, and PG perpendicular to Ad, we ſhall have, by putting PG $= a$, FG $= c$, AG $= n$, and DG $= x$; DF $= x - c$, AD $= n + x$, and conſequently, PD $= \sqrt{a^2 + x^2}$; whence, by ſimilar Triangles, DF : DP :: DA : DB $= \frac{n+x}{x-c}\sqrt{a^2+x^2}$; which, by the Conditions of the Problem, is to be a *Minimum*, and therefore the Fluxion thereof $\frac{\overline{c+n}\times\dot{x}\times\overline{a^2+x}+\overline{n+x}\times\overline{c-x}\times x\dot{x}}{\overline{c-x}\rvert^2\times\sqrt{a^2+x^2}}$ $= 0$; conſequently $x^3 - 2cx^2 - ncx = a^2 \times \overline{n+c}$; from whence x may be determined, and conſequently the Length of the Line (DB) paſſing through the given Point.

The ſame anſwered. R.

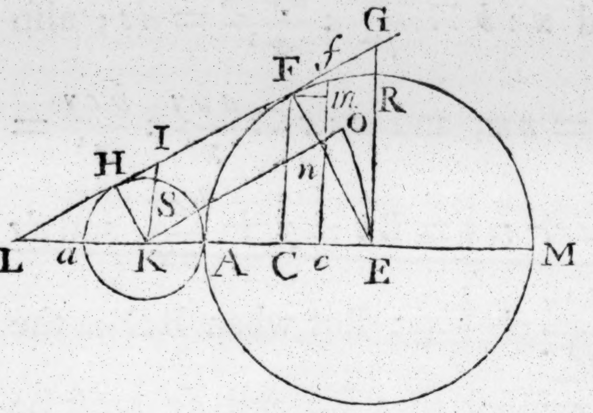
Let the Extremities of the two right Lines DB, db paſſing through the given Point P, and terminating in the two right Lines given by Poſition, be ſuppoſed indefinitely near to each other, and let DS and bm be ſo drawn, that each of the Angles dsD, bmB may be equal to the given Angle DAB; alſo, let Dr and bn be drawn perpendicular to dP and BP reſpectively, and let the Poſition of the other Lines, in the Figure, remain as before: Then if PF be put $= a$, PC $= b$, FG $= c$, BC $= x$, and B$m = y$; by ſimilar Triangles, it will be, $x : b :: y : \frac{by}{x} = bm$; BP : DP :: $x : a :: \frac{by}{x}$

$: \frac{aby}{x^2} = Ds$, and $x : b :: \frac{aby}{x^2} : \frac{ab^2y}{x^3} = ds$; alſo $a : c :: \frac{by}{x} : \frac{bcy}{ax} = nm$, and $a : c :: \frac{aby}{x^2} : \frac{bcy}{x^2} = rs$; therefore $Bm + mn = Bn = y + \frac{bcy}{ax}$, and $ds + sr = dr = \frac{ab^2y}{x^3} + \frac{bcy}{x^2}$: But when BD is the *Minimum*, it is manifeſt that Bn and dr muſt be equal; therefore $x^3 + \frac{bcx^2}{a} = cbx + ab^2$; whence it appears, if AS be perpendicular to AD, that $x^3 = ab^2$; becauſe $c = FG$ is then $= 0$.

PROBLEM XVIII. *Anſwered.* R.

In order to which, it may not, perhaps, be improper to ſhew the Manner of inveſtigating the two following Series, as the former of them is applied to the Solution here propoſed. If the Radius of a Circle be r, and any Arch thereof z; the Sine correſponding to the ſame will be expreſſed by $z - \frac{z^3}{2 \cdot 3 r^2} + \frac{z^5}{2 \cdot 3 \cdot 4 \cdot 5 r^4} - \frac{z^7}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 r^6}$, *&c.* and the Coſine by $r - \frac{z^2}{2r} + \frac{z^4}{2 \cdot 3 \cdot 4 r^3} - \frac{z^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 r^5}$, *&c.*

For, let the Radius $FE = r$, Sine $FC = x$, Verſed-ſine $AC = y$, and the Arch $AF = z$; then, by ſimilar Triangles, $x : r :: \dot{y} : \dot{z}$, and $r - y : r :: \dot{x} : \dot{z}$; whence, from the former Proportion, $x\dot{z} = r\dot{y}$, and,



and, from the latter, $r\dot{z}-y\dot{z}=r\dot{x}$: But to obtain x in Terms of z and known Quantities, let x be aſſumed $=Az+Bz^3+Cz^5+Dz^7$, *&c.* and by Subſtitution $r\dot{y}=Az\dot{z}+Bz^3\dot{z}+Cz^5\dot{z}+Dz^7\dot{z}$, &c. alſo $r\dot{z}-y\dot{z}=rA\dot{z}+3rBz^2\dot{z}+5rCz^4\dot{z}+7rDz^6\dot{z}$, &c. whence, from the former of theſe Equations, $y=\frac{Az^2}{2r}+\frac{Bz^4}{4r}+\frac{Cz^6}{6r}+\frac{Dz^8}{8r}$, &c. and, from the latter, $r-y=rA+3rBz^2+5rCz^4+7rDz^6$, &c. conſequently $r-y=r-\frac{Az^2}{2r}-\frac{Bz^4}{4r}-\frac{Cz^6}{6r}-\frac{Dz^8}{8r}$, &c. $=rA+3rBz^2+5rCz^4+7rDz^6$, &c. Now, ſince the Homology of the Terms depend not at all on the Coefficients, but altogether on the Powers of the variable Quantity z; the reſpective Terms of the two laſt Series may be compared with each other, and from thence will be had $A=1$, $B=\frac{-1}{2.3r^2}$, $C=\frac{1}{2.3.4.5r^4}$, $D=\frac{-1}{2.3.4.5.6.7r^6}$, &c. which being ſubſtituted in the Series aſſumed above, in Place of A, B, C, D, *&c.* there will reſult that expreſſing the Sine, and being ſubſtituted in either of the Series proved $=r-y$, there will reſult that expreſſing the Coſine. Q. E. D.

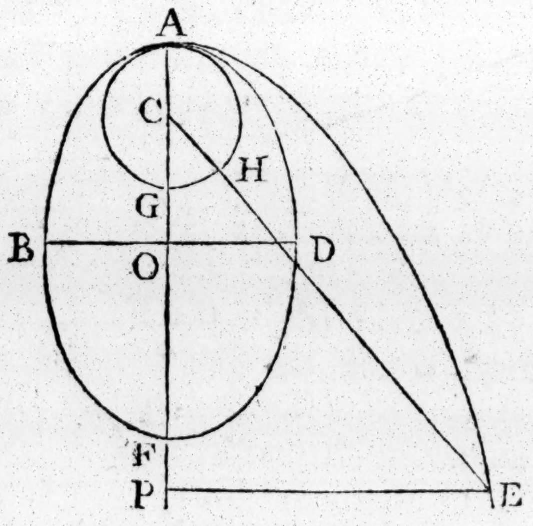
This being premiſed, if half the Length of the mixed Line be put $= S$, half the Sum of the two Diameters $= r$, the Arch $EO = z$, and the Sine correſponding the ſame $= x$; then will $Kn = HF = \sqrt{r^2 - x^2}$, $FE = \frac{r + x}{2}$, and $HK = \frac{r - x}{2}$; therefore $r : z :: \frac{r + x}{2} : \frac{rz + zx}{2r} = FR$, and $1 : \frac{p}{4} :: \frac{r + x}{2} : \frac{pr + px}{8} = MR$, p denoting the Periphery of a Circle whoſe Semi-diameter is Unity: In the ſame Manner it will appear that $HS = \frac{rz - zx}{2r}$, and $aS = \frac{pr - px}{8}$; conſequently $aH + MF + FH = \frac{pr}{4} + \frac{zx}{r} + \sqrt{r^2 - z^2} = S$; but x, by what has been already proved, is $= z - \frac{z^3}{2 . 3 r^2} + \frac{z^5}{2 . 3 . 4 . 5 r^4} - \frac{z^7}{2 . 3 . 4 . 5 . 6 . 7 r^6}$, &c. which being ſubſtituted above in Place of x, there will from thence ariſe an Equation, involving one unknown Quantity, whereby the Value of z may be determined, and from thence the Diameter of each Circle.

Problem XIX. *Anſwered by the* Propoſer.

The Equation defining the Curve, ſhould have been $y = \frac{\overline{ax - x^2}\rceil^{\frac{3}{2}} \times a^5 x^2}{\overline{a + x}\rceil^9}$: Then the given Value of y, by converting the Denominator to a Series, &c. may be transformed to $\frac{\overline{a - x}\rceil^{\frac{3}{2}} \times x^{\frac{1}{2}}}{a^4} \times \overline{x^3 - \frac{9 x^4}{a} + \frac{9 . 10 . x^5}{1 . 2 . a^2} - \frac{9 . 10 . 11 . x^6}{1 . 2 . 3 . a^3}}$, &c. where if the Fluent of $\overline{ax - x^2}\rceil^{\frac{1}{2}} \times \dot{x}$, when $x = a$, or the Area of the Semi-circle whoſe Diameter is a, be denoted by Q, and the whole Series be ſuppoſed to be multiplied by $\dot{x}$, the Fluent of the firſt Term will be $\frac{3.5.7.3.Q}{6.8.10.12}$; of the ſecond $-\frac{9.3.5.7.9.3.Q}{6.8.10.12.14}$; of the third $\frac{9.10.3.5.7.9.11.3.Q}{1.2.6.8.10.12.14.16}$, &c. and conſequently the Fluent of the whole Expreſſion will be $\frac{3.5.7.3.Q}{6.8.10.12}$ into $1-\frac{9.9}{14}+\frac{9.10.9.11}{1.2.14.16}-\frac{9.10.11.9.11.13}{1.2.3.14.16.18}$, &c. But the Sum of the Series $1-\frac{9.9}{14}+\frac{9.10.9.11}{1.2.14.16}$, &c. (by Prop. 1. *Summa. Series*, *Simpſon*'s Diſſertations) is found to be $\frac{419}{14336\sqrt{2}}$, which multiplied by $\frac{3.5.7.3.Q}{6.8.10.12}$ gives $\frac{7Q}{128}\times\frac{419}{14336\sqrt{2}}$ for the Area of the whole Curve, which therefore is to the Area of the Semicircle as 419 to $262144\sqrt{2}$.

PROBLEM XX. *Anſwered by Mr.* John Turner.

Let AHGA be the Earth, ADFBA the Trajectory, which the Body in Latitude 52° deſcribes, AEC that deſcribed by the Body projected from under the Equator, and A the Place where the Bodies leave the Earth's Surface. It is found (in Page 148 of *Simpſon*'s Fluxions) that the periodic Time of a Body deſcribing a Circle, juſt above the Earth's Surface, by Means of its own Gravity, is 84′ 43″, therefore the Velocity with which the Body at the Equator is projected, is to the ſaid circular Velocity, as $\frac{84' \, 43''}{38'}$, to 1, and therefore the Velocity of the other Body, being to the Velocity of the firſt, as the Coſine of (52°) the given Latitude to Radius, it will be expreſſed by $\frac{84' \, 43''}{38'} \times \frac{\text{Coſ. } 52°}{\text{Rad.}}$, the circular Velocity being denoted by Unity: Now let $\frac{84' \, 43'}{38'}$ be denoted by n, $\frac{84' \, 43''}{38'} \times \frac{\text{Coſ. } 52°}{\text{Rad.}}$ by m, and the Earth's Radius AC by a, then m^2 being leſs than 2, the Body projected from the Parallel of 52° will deſcribe an Ellipſis ABFDA, whereof the Semi-tranſverſe AO $= \frac{a}{2 - m^2}$, and the Semi-conjugate BO $= \frac{ma}{\sqrt{2 - m^2}}$: Therefore, of different Bodies revolving round the ſame Center, the Squares of their periodic Times, being as the Cubes of the tranſverſe Diameters, of the Section deſcribed, let the other Diameters be what they will, $AC^3 : AO^3 :: \overline{84' \, 43''}|^2 : \text{to } \frac{\overline{84' \, 43''}|^2}{\overline{2 - m^2}|^3}$ the Square of the periodic Time in the Ellipſis ABFDA; therefore $\frac{84' \, 43''}{\overline{2 - m^2}|^{\frac{3}{2}}}$ is the periodic Time required, which divided by 38′ leaves $\frac{84' \, 43''}{38' \times \overline{2 - m^2}|^{\frac{3}{2}}}$; therefore as 38′ is to the ſaid Remainder, ſo is 360° to the Difference of Longitudes of the two Points of the ſame Parallel from whence the Body is projected and where it falls.



Again,

Again, with regard to the other Body projected from under the Equator, the Curve AE which it deſcribes, becauſe n^2 is greater than 2, will be an Hyperbola whoſe Semi-tranſverſe is $\frac{a}{n^2-2}$, and the Semi-conjugate $\frac{na}{\overline{\sqrt{n^2-2}}}$; (ſee *Simpſon*'s Fluxions, Page 154) and the Hyperbolical Sector ACEA deſcribed in 6 Hours, will be to the Area of the Circle AHGA deſcribed in 38′, as 6 Hours to 38′, and therefore will be expreſſed by $\frac{\text{AHGA}\times 360'}{38'}$ $=\frac{a^2\times 360'\times 3.141592,\ \&c.}{38'}=\frac{ra^2}{2}$; but the Area ACEA, ſuppoſing the Ordinate PE to be expreſſed by az, will be alſo expreſſed by $\frac{\overline{n^2-1}\times a^2z}{2\times\overline{n^2-2}}$ $-\overline{\frac{na^2}{2\times n^2-2}}\big|^{\frac{3}{2}}\times$ Hip. Log. $\overline{\sqrt{a^2z^2+\frac{n^2a^2}{n^2-2}}+az}$ $\times\frac{\sqrt{n^2-2}}{na}=\frac{ra^2}{2}$, or $\frac{\overline{n^2-1}\times z}{n^2-2}-\overline{\frac{n}{n^2-2}}\big|^{\frac{3}{2}}\times$ Hip. Log. $\overline{\sqrt{z^2+\frac{n^2}{n^2-2}}+z}\times\frac{\sqrt{n^2-2}}{n}=r$; therefore, putting $n^2-1=b$, $\frac{n}{\overline{\sqrt{n^2-2}}}=c$, and $\frac{1}{n^2-2}=d$, it will become $bdz-cd\times$ Hip. Log. $\frac{\overline{\sqrt{z^2+c^2}+z}}{c}=r$, or $z=\frac{r}{bd}+\frac{c}{b}\times$ Hip. Log. $\frac{\sqrt{z^2+c^2}+z}{c}$; whence the Diſtance of the Body from the Earth's Center may be determined.

This Problem was, alſo, anſwered by *Tycho Oxonienſis*, who makes the periodic Time of the Body projected from the Parallel of 52°, to be 1 Day 9 Hours 44′ 52″, and that projected from under the Equator, to be 186947 Miles diſtant from the Earth's Center at the End of the propoſed Time.

A
COLLECTION
OF
PROBLEMS
TO BE
Anſwered in the next NUMBER.

PROBLEM XXI. *L.*

IF to enjoy the Benefit of an Eſtate for 23 Years after the Expiration of eight Years, be worth 400*l.* preſent Money; what will the ſaid Eſtate be worth for 21 Years after the Expiration of 10 Years, at the Rate of 5*l.* for every 100 Yearly.

PROBLEM XXII. *L.*

A Traveller benighted, ſees before him two Lights, and looking back diſcovers three others, in a right Line with the former; and judges the Quantity of Light received from the former Place, to be to that received from the latter, in the Ratio of 3 to 4; then proceeding forwards 400 Yards, he finds the Ratio of Light there to become as 5 to 3: The Queſtion is, ſuppoſing all the Lights to be equal among themſelves, what Diſtance he was from each Set of Lights, at the two Places of Obſervation.

PROBLEM XXIII. *A* Theorem *to be demonſtrated.*

As the leſſer of the two Sides, including any propoſed Angle, of a Triangle, is to the greater, ſo is Radius, to the Tangent of an Arch or Angle: And as Radius, is to the Tangent of the Exceſs of the ſaid Angle above half a right Angle, ſo is the Tangent of half the Sum of the oppoſite Angles, to the Tangent of half the Difference of the ſame Angles.

PROBLEM XXIV. *Thomas Hulme*, London.

The Line biſecting the Baſe, the Difference of the Sides, and the Difference of the Angles at the Baſe, of any plane Triangle being given; to determine the Triangle.

PROBLEM XXV. *John Turner*, London.

The Baſe of any plane Triangle, the vertical Angle and the Side of the inſcribed Square being given; to conſtruct the Triangle.

PROBLEM XXVI. *R.*

To deſcribe a Circle, through a given Point, which ſhall touch a right Line given in Poſition, and alſo another Circle given in Magnitude and Poſition.

PROBLEM XXVII. *R.*

To deſcribe a Circle, through two given Points, that ſhall touch another Circle given in Magnitude and Poſition.

PROBLEM XXVIII. *N.*

One Side and all the Angles, made by the adjacent Sides and the two Diagonals, of any Trapezium being given, except the two Angles contained between them and the given Side; to deſcribe the Trapezium: And that without aſſuming any ſimilar Figure.

PROBLEM XXIX. *John Moor.*

One Side and the Diſtances from the Center of Gravity to each Angle, of any Trapezium being given; to determine the Trapezium.

PROBLEM XXX. *Walter Trott*, London.

A Ship ſailed from the *Lizard* (Lat. 50° 00′ N.) S. W. 20 Miles, and was there taken by a *French* Privateer; who took away their Compaſs, and afterwards kept them Company ſteering between the South and Eaſt until, as the Privateer told them, the *Lizard* bore due North, and then the Privateer left them; and they continued the ſame Courſe, as near as poſſible, and run by the Log. 25 Miles; then they ſpoke with a Man of War, who informed them the *Lizard* bore N. W. by W. required the Courſe and Diſtance ſailed in Company with the Privateer; alſo their Diſtance from the *Lizard*, with the Latitude the Ship is in, and her Departure from the Meridian.

PROBLEM XXXI. *John Turner*, London.

Of all the ſpheroidical Caſks having the ſame given Diagonal, to find that whoſe Content is the greateſt.

PROBLEM XXXII. *R.*

Of all the conic Parabolas paſſing through four given Points, to determine the leaſt.

PROBLEM XXXIII. *Hamilcar.*

To determine ſuch a Part of a ſpherical Superficies, which can be illuminated in its farther Part, by Light coming from a great Diſtance, and which is refracted by the nearer Hemiſphere.

PROBLEM XXXIV. *R.*

If a Curve be ſuppoſed to revolve upon its principal Axe, and thereby generate a Solid, the Fruſtum of which is of ſuch a Nature, that the Difference between the two Diameters multiplied by any fractional Number leſs than Unity, and the Reſult taken from the greater, produces the Diameter of a Cylinder of equal Magnitude having the ſame Length as the Fruſtum; then it is required to find the Equation of the Curve from whence it is generated.

PROBLEM XXXV. *T. G——t.*

If the Relation between the Abſciſs, and the Ordinate of a Curve, be expreſſed by $yx = \overline{a + cx^n}\rceil^m \times dx^{pn}$; the Area, ſuppoſing x the Abſciſs, y the Ordinate, and $a + cx^n = v$, will be expreſſed by $\frac{dv^m x^{pn}}{pn} \times \overline{1 - \frac{m}{p+1} \times \frac{cx^n}{v} + \frac{m.m-1}{p+1.p+2} \times \frac{c^2 x^{2n}}{v^2} - \frac{m.m-1.m-2}{p+1.p+2.p+3} \times \frac{c^3 x^{3n}}{v^3}}$, *&c.* Required the Inveſtigation.

PROBLEM XXXVI. *R.*

Let it be required to find the Sum of the Series $\frac{m-1.n-1.p.-1.}{r} + \frac{m-2.n-2.p-2}{r^2} + \frac{m-3.n-3.p-3}{r^3}$, *&c.* continued to m Terms, and to ſhew the Inveſtigation of the ſame.

PROBLEM XXXVII. *Thomas Leigh*, London.

If a Ciſtern, in Form of the Fruſtum of a Sphere, whoſe Axis, or Depth is 10 Feet, and the Diameter at the Top 30 Feet, be ſupplied with Water running uniformly from a Cock that can fill it in two Hours, and at the Bottom be another Cock, out of which the whole Ciſtern may be evacuated in one Hour, the former Cock being ſtopped; it is required to find, if the Ciſtern was full, and both the ſaid Cocks open, in what Time the Surface would deſcend four Feet, and the greateſt Diſtance it could poſſibly deſcend.

PROBLEM XXXVIII. *R.*

To find, ſuppoſing the Earth's Rotation about its Axis was entirely to ceaſe, how much Pendulums would gain in 24 Hours in the Latitude 51° 30′.

PROBLEM XXXIX. *John Turner*, London.

Suppoſe the Earth, inſtead of being nearly ſpherical, was to revolve about its Axis, in ſuch a Time, that the Equatorial Diameter ſhould be juſt double the Polar Diameter; it is propoſed to find in what Time, a Body in the Latitude 50 Degrees, falling every where in the Direction in which it gravitates without Interruption or Reſiſtance, would deſcend to the Earth's Center.

PROBLEM XL. *T.*

To inveſtigate the Path, which a fixed Star by Means of the Aberration would appear to deſcribe, if the Earth inſtead of revolving in an Ellipſis, was to move in a Parabolic or Hyperbolic Orbit.

The End of NUMBER II.